CHAPTER 2

Waves in an Isotropic Elastic Solid

Suppose that all the particles within a gas or a fluid or a solid are stationary inside a volume V, but then undergo small internal displacements $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ in response to applied forces or to changes in the surface forces (tractions) applied on the surface S of the volume.

Let $\mathbf{f}$ be the applied force, acting (per unit volume) on particles inside V. $\mathbf{f}$ is a function of space and time: $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$. Examples of body forces, are gravitational attraction, and magnetic attraction or repulsion. Earthquake sources inside V may also be represented by body forces, as can an instructor speaking in a classroom and sending out sound waves through the air.

The rate of change of momentum of particles making up V equals the forces acting on these particles:

$$\frac{\partial}{\partial t}\iiint_V \rho \frac{\partial \mathbf{u}}{\partial t} dV = \iiint_V \mathbf{f}\, dV + \iint_S \mathbf{T}(\mathbf{n})\, dS. \tag{2.1}$$

Here, the volume V and surface S move with the particles, we are using a Lagrangian description of motion, the left-hand side is $\iiint_V \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} dV$ since the particl mass ρdV is not changing with time, and $\mathbf{T}$ is related to $\mathbf{n}$ by the rules $T_i = \tau_{ji} n_j = \tau_{ij} n_j$ ($\boldsymbol{\tau}$ is symmetric).

But

$$\iint_S A n_j dS = \iiint_V \frac{\partial A}{\partial x_j} dV \tag{2.2}$$

for any differentiable quantity $A = A(\mathbf{x})$ (see Box 2.1).
So

$$\iint_S T_i\, dS = \iint_S \tau_{ij} n_j\, dS = \iiint_V \frac{\partial \tau_{ij}}{\partial x_j}\, dV = \iiint_V \tau_{ij,j}\, dV.$$

It follows that we can convert all the terms in (2.1) to volume integrals, and put them on the left-hand side as

$$\iiint_V (\rho \ddot{u}_i - f_i - \tau_{ij,j}) dV = 0. \tag{2.3}$$

Because this result is true for all volumes V, this integrand must be zero wherever it is

BOX **2.1**

Generalization of Gauss's Divergence Theorem

Gauss's divergence theorem is

$$\iint_S \mathbf{a} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{a}\, dV \tag{1}$$

for vector **a**. Writing this out as

$$\iint_S (a_1 n_1 + a_2 n_2 + a_3 n_3)\, dS = \iiint_V \left(\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} \right) dV, \tag{2}$$

this result is actually valid "term-by-term,"

$$\iint_S a_1 n_1\, dS = \iiint_V \frac{\partial a_1}{\partial x_1}\, dV, \quad \text{etc.} \tag{3}$$

The proof of all these results, (1), (2), (3), is based essentially on

$$f(q) - f(p) = \int_{f(p)}^{f(q)} df = \int_p^q \frac{df}{dx}\, dx.$$

continuous (otherwise, we could surround a place where it is not zero with a small volume that violates (2.3)). Hence

$$\rho \ddot{u}_i = f_i + \tau_{ij,j}, \tag{2.4}$$

which is our first form for an equation of motion. We have not yet made any assumptions about the relations between stress and strain, so (2.4) is a very general result. Essentially, it says that particles within a moving deformable body have a rate of change of momentum (mass × acceleration) that is driven by the applied body force plus the stress gradient.

2.1 Compressional and shearing motions in an isotropic elastic medium

For an isotropic elastic medium, the stress tensor $\boldsymbol{\tau}$ and the strain tensor **e** have components that are related by the generalized version of Hooke's Law:

$$\tau_{ij} = \lambda e_{k,k} \delta_{ij} + 2\mu e_{ij}. \tag{1.37 again}$$

Combining this result with (2.4) and the definition of strain given by (1.29), we get the displacement equation for elastic motion in an isotropic medium. The equation can take a number of different forms, all equivalent. Specifically, if the medium is homogeneous so that we do not have gradients of the Lamé moduli,

$$\begin{aligned} \rho \ddot{u}_i &= f_i + \lambda u_{k,kj} \delta_{ij} + \mu (u_{i,jj} + u_{j,ij}) \\ \rho \ddot{u}_i &= f_i + (\lambda + \mu) u_{j,ji} + \mu u_{i,jj} \end{aligned} \tag{2.5}$$

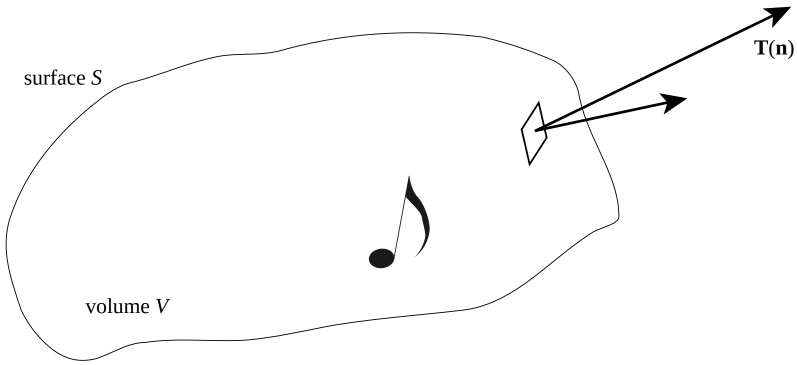


FIGURE **2.1**
An elastic solid with volume V and surface S is subjected to applied tractions. Body forces $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ may act inside V, for example at a point where someone is singing or talking and leading to sound waves that spread throughout the volume.

or (using vectors rather than vector components)

$$\rho\ddot{\mathbf{u}} = \mathbf{f} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2\mathbf{u}.$$

Using the definition $\nabla^2\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$, this gives

$$\begin{aligned}\rho\ddot{\mathbf{u}} &= \mathbf{f} + (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times (\nabla \times \mathbf{u}), \text{ or} \\ \rho\ddot{\mathbf{u}} &= \mathbf{f} + (\lambda + 2\mu)\,\text{grad}\,\text{div}\mathbf{u} - \mu\,\text{curl}\,\text{curl}\,\mathbf{u}.\end{aligned} \tag{2.6}$$

All of these different forms of the elastic wave equation are equally valid. But how do they compare with simpler wave equations? We shall find that our elastic wave equation permits two completely different types of solution — P-waves and S-waves.

First let's find what happens if we try to find a plane wave propagating with speed c in the x_1-direction in the absence of applied forces ($\mathbf{f} = \mathbf{0}$). That is, we try for a solution to (2.5) or (2.6) in the form

$$\mathbf{u} = \mathbf{u}\left(t - \frac{x_1}{c}\right). \tag{2.7}$$

Here we are also assuming that the dependence on position is via the x_1-coordinate alone, so

$$\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} \quad \text{and} \quad \nabla \times \mathbf{u} = \left(0, -\frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1}\right).$$

From the wave equation (2.5) or (2.6) we see that

$$\begin{aligned}\rho\ddot{u}_1 &= 0 + (\lambda + \mu)u_{1,11} + \mu u_{1,11} = (\lambda + 2\mu)u_{1,11}\\ \rho\ddot{u}_2 &= 0 + 0 + \mu u_{2,11} = \mu u_{2,11}\\ \rho\ddot{u}_3 &= 0 + 0 + \mu u_{3,11} = \mu u_{3,11}.\end{aligned} \tag{2.8}$$

The three scalar equations in (2.8) are each examples of a one-dimensional (1D) wave equation, of the type discussed in Box 2.2. By comparison with the 1D solution discussed in this Box we see that the u_1 component of motion in (2.8) propagates as a wave having speed $c = \sqrt{\frac{\lambda + 2\mu}{\rho}}$. This is the P-wave (the P standing for *primary*, because this the fastest traveling wave in an elastic solid). The particle motion is in the same direction as the direction of propagation. A wave with this property is said to be *longitudinal*. From the second and third of the equations in (2.8), which are also 1D wave equations, we see that the u_2 and u_3 components of motion have speed $c = \sqrt{\frac{\mu}{\rho}}$. These are examples of the S-wave (the S standing for *secondary*). Because the particle motion is now perpendicular to the direction of propagation (x_1), these solutions for u_2 and u_3 are each examples of a *transverse* wave.

2.1.1 SIMPLE EXAMPLES OF 1D WAVE PROPAGATION

We can illustrate several basic properties of P-waves and S-waves with two simple examples that have a lot in common with plane waves traveling in three dimensions (3D).

(i) *The longitudinal wave in a spring (slinky).* Stretch the spring or slinky on the surface of a smooth table, and tap the end to initiate a longitudinal (P-wave) motion that travels down the spring. If the spring has mass m per unit length, then an element δx of the spring has a rate of change of momentum given by $(m\ \delta x)\ddot{u}$, where u is the displacement in the direction of the spring (a longitudingal motion). The total applied force is given by the difference in tension in the spring at each end of the line element, $T(x + \delta x) - T(x)$. But if the original string tension was T_0 then the new tension is $T(x) = T_0 + k\frac{\partial u}{\partial x}$ where k is related to Young's modulus (see Problem 1.10: $k = E \times$ cross-sectional area of the spring). It follows that

$$m\frac{\partial^2 u}{\partial t^2} = \frac{\partial T}{\partial x} = k\frac{\partial^2 u}{\partial x^2},$$

a 1D wave equation with speed $c = \sqrt{\frac{k}{m}}$.

(ii) *The transverse wave in a stretched string or rope.* Take a rope, preferably a few tens of meters in length, and tie it to a support at each end so that the rope is approximately horizontal and tightly stretched. Tap the rope near one end in a direction that is (a) horizontal, and (b) perpendicular to the rope. A transverse wave of horizontal motion travels the length of the rope, and the wave may be reflected at the ends (if the set-up is working well), so that the wave goes back and forth a few times before it is attenuated (due to friction in the rope fibers). In this case, let u be the displacement of the rope, in the horizontal direction perpendicular to the rope (parallel to the direction

BOX **2.2**

On the most general solution of the simplest second-order wave equation

The simplest second-order wave equation for the variable ϕ is

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}, \tag{1}$$

in which there is a dependence upon only one spatial dimension (x), and time (t).

To obtain the most general solution of (1), for ϕ as a function of space and time, we define two new independent variables:

$$\xi = t - x/c \qquad \text{and} \qquad \eta = t + x/c. \tag{2}$$

Then

$$\frac{\partial\phi}{\partial x} = \frac{\partial\xi}{\partial x}\frac{\partial\phi}{\partial\xi} + \frac{\partial\eta}{\partial x}\frac{\partial\phi}{\partial\eta} = -\frac{1}{c}\frac{\partial\phi}{\partial\xi} + \frac{1}{c}\frac{\partial\phi}{\partial\eta} = -\frac{1}{c}\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right)\phi \qquad \text{and}$$

$$\frac{1}{c}\frac{\partial\phi}{\partial t} = \frac{1}{c}\frac{\partial\xi}{\partial t}\frac{\partial\phi}{\partial\xi} + \frac{1}{c}\frac{\partial\eta}{\partial t}\frac{\partial\phi}{\partial\eta} = \frac{1}{c}\frac{\partial\phi}{\partial\xi} + \frac{1}{c}\frac{\partial\phi}{\partial\eta} = \frac{1}{c}\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\phi.$$

So

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right)\left(\frac{\partial\phi}{\partial\xi} - \frac{\partial\phi}{\partial\eta}\right) = \frac{1}{c^2}\left(\frac{\partial^2\phi}{\partial\xi^2} - 2\frac{\partial^2\phi}{\partial\xi\,\partial\eta} + \frac{\partial^2\phi}{\partial\eta^2}\right) \qquad \text{and}$$

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{1}{c^2}\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\left(\frac{\partial\phi}{\partial\xi} + \frac{\partial\phi}{\partial\eta}\right) = \frac{1}{c^2}\left(\frac{\partial^2\phi}{\partial\xi^2} + 2\frac{\partial^2\phi}{\partial\xi\,\partial\eta} + \frac{\partial^2\phi}{\partial\eta^2}\right).$$

The wave equation (1) therefore becomes

$$0 = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial\phi}{\partial x^2} = 4\frac{\partial^2\phi}{\partial\xi\ \eta}.$$

But we can easily solve

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\phi}{\partial\eta}\right) = 0 \tag{3}$$

as follows. First we integrate with respect to ξ to obtain $\frac{\partial\phi}{\partial\eta} = g(\eta)$ for some function g. (This result is just another way of saying that $\frac{\partial\phi}{\partial\eta}$ cannot depend upon ξ — which is what we learn from equation (3) — and therefore can depend only upon η.) Second we integrate again, this time with respect to η, to get

$$\begin{aligned}\phi &= F(\xi) + \int^{\eta} g(\eta')\,d\eta' \\ &= F(\xi) + G(\eta) \\ &= F(x - ct) + G(x + ct).\end{aligned} \tag{4}$$

The wave equation (1) therefore indicates that ϕ must be a function of $t - x/c$ plus a function of $t + x/c$. This is the most general form of solution of the simplest wave equation in one dimension, equation (1).

BOX **2.2** (continued)

When we integrate a second-order *ordinary* differential equation we can expect to find that the solution in general contains two arbitrary *constants*. Here we have integrated a second-order *partial* differential equation, and have found that the solution (4) in general contains two arbitrary *functions*, of space and time in particular combinations (either $t - x/c$, or $t + x/c$).

The wave solution itself can have any shape. (F or G are any sufficiently smooth functions — though shortly we shall discuss the need for them to have some type of discontuity.) Note that the wave solution of (1) propagates without change in shape (either F propagating in one direction, or G propagating in the opposite direction). What matters, are the particular combinations of space and time upon which F and G depend.

Another way to approach the solution of (1), is to ask if there is a more general combination of space and time, such as $t - T(x)$, upon which solutions ϕ might depend. Substituting $\phi = \phi(t - T(x))$ into (1) and using $\dfrac{\partial\phi}{\partial x} = -\dfrac{dT}{dx}\dfrac{\partial\phi}{\partial t}$, we find

$$\left(\left(\frac{dT}{dx}\right)^2 - \frac{1}{c^2}\right)\frac{\partial^2\phi}{\partial t^2} - \frac{d^2T}{dx^2}\frac{\partial\phi}{\partial t} = 0. \tag{5}$$

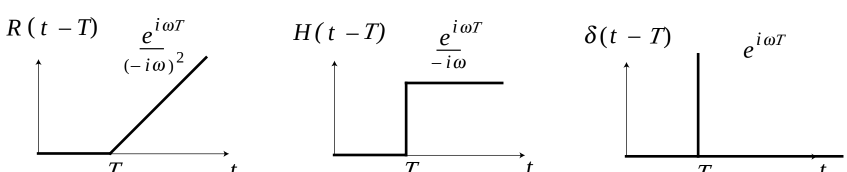
If we think of a wave as a solution of (1) that is able to propagate non-zero values of some physical variable (represented by ϕ) into a region where previously $\phi = 0$, then we need to consider solutions $\phi(x, t)$ that contain a discontinuity in some derivative of ϕ.

[If $\phi = 0$ inside a region S of space and time, then all the derivatives of ϕ in that region are zero also. But if there is no discontinuity anywhere in any derivatives of ϕ, then the Taylor series expansion

$$\phi(x + \delta x, t + \delta t) = \phi(x, t) + \delta x\frac{\partial\phi}{\partial x} + \delta t\frac{\partial\phi}{\partial t} + \frac{\delta x^2}{2}\frac{\partial^2\phi}{\partial x^2} + 2\,\frac{\delta x\ \delta t}{2}\,\frac{\partial^2\phi}{\partial x\,\partial t} + \frac{\partial t^2}{2}\frac{\partial^2\phi}{\partial t^2} + \ldots$$

implies that $\phi(x + \delta x, t + \delta t) = 0$ if (x, t) is taken within S. But then the region S has been expanded to include $(x + \delta x, t + \delta t)$. All derivatives of ϕ are zero in the larger region. In this situation it is impossible to bound the region in which ϕ is zero. The only way to obtain non-zero values is therefore to require that some derivative of ϕ is non-zero.]

The Figures (a)–(c) here show an example where $\phi(t - T)$ is the ramp function $R(t - T)$, given by $R(t) = 0$ if $t \le 0$ and $R(t) = t$ if $0 \le t$. Then $\dfrac{\partial\phi}{\partial t}$ is the Heaviside step function $H(t - T)$ ($H(t) = 0$ if $t < 0$, $H(t) = 1$ if $0 < t$), and $\dfrac{\partial^2\phi}{\partial t^2}$ is the Dirac delta function $\delta(t - T)$ ($\delta(t) = 0$ if $t \ne 0$, but the area under $\delta(t)$ for a range of t-values where $t = 0$, is unity). The Fourier spectrum of each function is also given in these Figures.



(a) (b) (c)

We define a *wavefront* as a propagating discontinuity in the solution to a wave equation. Thus the wavefronts for our simplest wave equation, (1), can be determined as the propagating surfaces $t = T(x)$ in (x, t) space, across which some derivative of ϕ has a discontinuity. In the case shown above with Figures (a)–(c), we can use equation (5) and integrate the left-hand side across a small region from $t = T - \varepsilon$, to $t = T + \varepsilon$. The last term in (5) integrates to a result proportional to ε, which is zero in the limit as $\varepsilon \to 0$.

BOX **2.2** (continued)

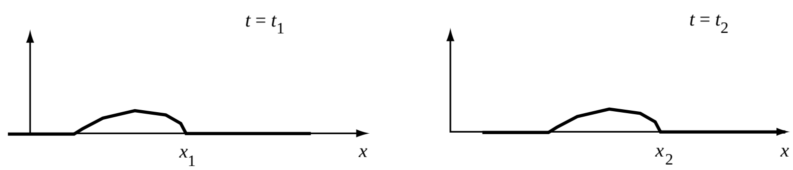
But the term $\frac{\partial^2\phi}{\partial t^2}$ in (5) integrates to a non-zero value, so its coefficient must vanish and at last we obtain the result

$$\left(\frac{dT}{dx}\right)^2 - \frac{1}{c^2} = 0.$$

This is the equation which determines the wavefronts of the one-dimensional wave equation (1). In the present case the solutions are simple, for

$$\frac{dT}{dx} = \pm\frac{1}{c},$$

and hence $T = x/c$ or $T = -x/c$ (plus a constant, which will be zero if we arrange that $T = 0$ at the position $x = 0$). The relevant combinations of space and time upon which ϕ must depend are then seen to be those we chose in (2), and the interpretation of these alternative independent variables, $\xi = t - x/c$ and $\eta = t + x/c$, is that they are measures of distance from a wavefront. In practice, $T = T(x)$ is seen to be the travel time of the wave, that is, the time it takes for the wave to reach position x. Then $t - T(x)$ is the time after the wavefront arrival (so that negative values of $t - T$ give zero values of ϕ).



To see that c is the speed of propagation of the wave $\phi = F$, we can compare the spatial dependence of F at two different times $t = t_1$ and $t = t_2$ as shown in the Figure here. A particular feature of the wave is chosen — say, the spatial position at which it has values that begin to depart from $F = 0$. (Any feature that we can track as a function of space and time will serve as a satisfactory marker.) At $t = t_1$ this feature is at $x = x_1$, and at $t = t_2$ it is at $x = x_2$. Because this is the same feature of the moving wave $\phi = F(t - x/c)$, a wave that depends on space and time only in the combination $t - x/c$, we know that

$$t_1 - x_1/c = t_2 - x_2/c. \tag{7}$$

in which the rope was initially tapped). If the rope has mass m per unit length, then an element δx of the rope has a rate of change of momentum given by $m\ \delta x)\ddot{u}$. In this case, the original tension T_0 in the rope is unchanged because the transverse motion produces a change in rope length that is negligible (it is of order u^2). The Figure here shows the directions in which x and u are taken, and an enlarged view of the element of the rope between x and $x + \delta x$:

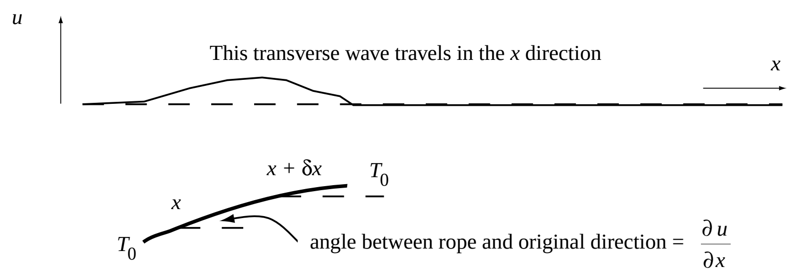
BOX **2.2** (continued)

In the time period $t_2 - t_1$ the feature has moved a distance $x_2 - x_1$, so that the speed of propagation is $(x_2 - x_1)/(t_2 - t_1)$. Equation (7) then tells us that this speed, a ratio of distance to time, is precisely c.

Finally, we may note that (1) is a combination of the simpler wave equations

$$\frac{1}{c}\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial x} \quad \text{and} \quad \frac{1}{c}\frac{\partial\phi}{\partial t} = -\frac{\partial\phi}{\partial x}. \tag{6}$$

The first of these first-order wave equations describes a wave moving with speed c in the x-direction, and the second a wave moving with speed $-c$ (that is, with speed $+c$ in the negative x-direction).



The force applied to the mass $m\ \delta x$ in the direction in which u is measured, is $T_0 \left.\frac{\partial u}{\partial x}\right|_{x+\delta x}$ at one end, and $-T_0 \left.\frac{\partial u}{\partial x}\right|_{x}$ at the other. So

$$m\ \delta x\ \frac{\partial^2 u}{\partial t^2} = T_0\ \left.\frac{\partial u}{\partial x}\right|_{x+\delta x} - T_0\ \left.\frac{\partial u}{\partial x}\right|_{x}$$

and the 1D wave equation is

$$m\frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2},$$

so that the speed is $c = \sqrt{\frac{T_0}{m}}$.

Instead of tapping the string horizontally, it could have been tapped in a direction that is still perpendicular to the rope, but in the vertical plane. This too is a way to initiate a transverse wave, that is independent of the horizontal transverse motions. In general the rope can support a transverse wave that is an arbitrary mix of these two possibilities. The *polarization* of the general wave, is a measure of the mix of horizontal and vertical motions, both being transverse to the direction of propagation.

If the ends of the rope are not at the same level then the rope itself is not

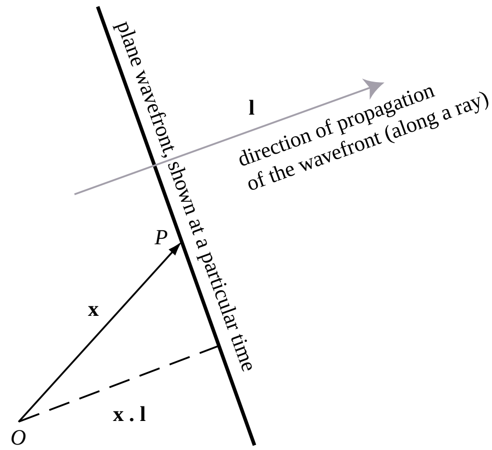


FIGURE **2.2**
The position of a plane wavefront is shown at a fixed time. P is a point at position **x** on the wavefront. For any such **x**, the value of $\mathbf{x} \cdot \mathbf{l}$ is the same since this scalar product is the perpendicular distance from the origin of coordinates at O to the wavefront (a distance which is independent of position **x** as long as it is in the wavefront).

horizontal. The first transverse motion we have discussed, above, can still be taken in the horizontal direction which is tranverse to the rope. Such a motion is called an SH-wave. The other transverse motion will be perpendicular to the rope, but if the rope is no longer horizontal it will not be a purely vertical motion — although it will lie in a vertical plane. Such a motion is called an SV-wave. The polarization of a general transverse wave, is a measure of how much of the transverse motion is SH, and how much is SV.

2.1.2 THE GENERAL PLANE WAVE IN AN ISOTROPIC ELASTIC MEDIUM

Here, we shall examine solutions of the 3D wave equations (2.6) that are very similar to those described by equation (2.7) and Section 2.1.1, but the propagation now is assumed to be in a general direction **l** (a unit vector). Some of the main results of this Section are left as an exercise (Problem 2.1).

First, we need to appreciate the defining property of a plane wave. A dependent variable (such as displacement **u** or a particular stress component such as τ_{23}) travels as a plane wave, if values of the variable are unchanged for any point on a moving planar surface. This planar surface propagates as a wavefront in the direction specified by the unit vector **l** (see Figure 2.2). The equation of all points lying on a plane perpendicular to **l** is $\mathbf{x} \cdot \mathbf{l} =$ constant. This constant, is just the perpendicular distance from the origin (where $\mathbf{x} = \mathbf{0}$) to the plane of interest. As the constant is changed from one value to another, a different plane is specified. All the planes are perpendicular to **l**. The constant is *zero*, for the plane perpendicular to **l** that also lies on the origin O itself. For a plane wavefront that moves with speed c perpendicular to itself, $\mathbf{x} \cdot \mathbf{l} = ct$. If we rewrite this as $t = T(\mathbf{x})$ then we see in this case that $T = \frac{\mathbf{x} \cdot \mathbf{l}}{c}$, which is the travel time.

A general plane wave solution to the elastic wave equation

$$\rho \ddot{\mathbf{u}} = (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}) \qquad \text{(equation(2.6) with no body force)}$$

can therefore be taken as

$$\mathbf{u} = \mathbf{u}\left(t - \frac{\mathbf{x} \cdot \mathbf{l}}{c}\right). \tag{2.9}$$

We can now generalize the result given in (2.8), in which one-dimensional elastic waves traveling in the x_1-direction were obtained (both P-waves and S-waves). In the present more general case of propagation in the **l** direction, with (as yet) no contraint on the direction of particle motion (the direction of **u** in (2.9)), we can substitute our new form of trial solution into (2.6). The details are best worked out by the reader (see Problem 2.1).

2.1.3 WAVEFRONTS AND RAYS

The plane wave shown in Figure 2.2 will move to a new position at a later time. Figure 2.3a shows the same plane wavefront at a number of different times. This system of wavefronts is an example of the wavefront equation $t = T(\mathbf{x})$ in which t is given five different values, and for each t value the set of **x** values solving $t = T(\mathbf{x}$ gives the position of the wavefront at that t value. In other words, these values of **x** all share the same travel time. As the travel time increases, the wavefront moves to a new position. Orthogonal to the moving wavefront, is the set of rays. In the case of a plane wavefront, the rays are parallel to each other.

Figure 2.3b shows a completely different wavefront, namely an expanding spherical wavefront. This too is a wavefront governed by the equation $t = T(\mathbf{x}$, but now a three-dimensional wave equation applies, for example

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi, \tag{2.10}$$

and a trial solution is sought in the form

$$\phi(\mathbf{x}, t) = A(\mathbf{x}) P\left((t - T(\mathbf{x})\right). \tag{2.11}$$

With this form of solution, again we seek a propagating discontinuity given by $t = T(\mathbf{x})$. The factor $P(t - T)$ represents the "pulse shape" of the wave, and the factor $A(\mathbf{x})$ governs the change is amplitude with different position. This factor was missing for the plane wave solutions we have looked at previously, because for them the wave propagated without change in amplitude. In the example of a spherical wave shown in Figure 2.3b, we would expect the amplitude to decrease as the wavefront expands. The term $A(\mathbf{x})$ in (2.11) is sometimes called the *geometrical spreading factor*. Usually, it is not possible to get an exact solution to (2.10) in the form (2.11). But this form of trial solution is adequate for finding the propagating discontinuities associated with the wave equation (2.10) (see the

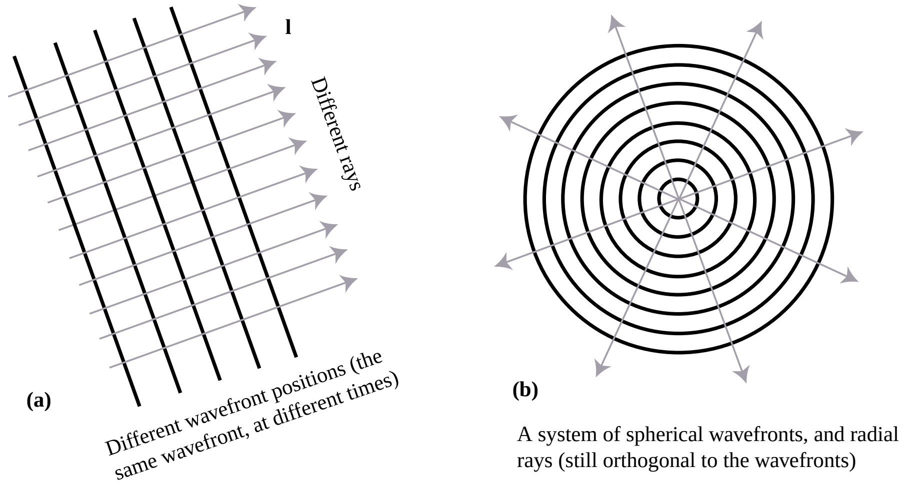


FIGURE **2.3**
(**a**) On the left, is shown a propagating plane wavefront. Its position is indicated at five different times. Orthogonal to the system of wavefronts, is the system of rays. In this plane-wave example, the direction **l** is constant along the ray, and has the same value for all rays. (**b**) On the right, is shown another system of wavefronts, in this case an expanding spherical wavefront shown at eight different times. Again the rays are an orthogonal system, and again these are straight line rays (because the medium os homogeneous, with a fixed value of the propagation speed c). But now the rays themselves are not parallel.

discussion in Box 2.3). Fortunately, in the case of a spherical wavefront expanding in a homogeneous isotropic medium, it is possible to find an exact solution to (2.10) in the form (2.11). Noting that the radial derivatives in $\nabla^2\phi$ are $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2(r\phi)}{\partial r^2}$, we see that the product $r\phi$ satisfies the one dimensional wave equation

$$\frac{1}{c^2}\frac{\partial^2(r\phi)}{\partial t^2} = \frac{\partial^2(r\phi)}{\partial r^2} \tag{2.12}$$

provided the solution is spherically symmetric (so that the spatial dependence in only on the radial coordinate).

From the knowledge of solutions to the 1D wave equation gained in Box 2.2, it follows that the solution of (2.12) has the form

$$\phi(\mathbf{x}, t) = \frac{1}{r}P\left(t - \frac{r}{c}\right). \tag{2.13}$$

This is an exact result, having the form of solution given in (2.11). The geometrical spreading factor is simply $\frac{1}{r}$. The wavefront equation, which in general takes the form $t = T(\mathbf{x})$, in this case is just $t = \frac{r}{c}$.
[Note to PGR: add here a discussion, with new Figure, showing wavefronts and rays in an example for which $c = c(\mathbf{x})$.]

BOX **2.3**

On finding approximate solutions to the 3D wave equation

If we substitute the trial solution (2.11) into the wave equation (2.10), we can get an exact solution if A, T, and P satisfy

$$\left((\nabla T)^2 - \frac{1}{c^2}\right)\ddot{P}(t-T) - \left(\nabla^2 T + 2\frac{\nabla A \cdot \nabla T}{A}\right)\dot{P}(t-T) + \frac{\nabla^2 A}{A}P(t-T) = 0. \quad (1)$$

If we seek a solution $\phi = A(\mathbf{x})P(t - T(\mathbf{x})$ that has a propagating discontinuity given by the wavefront equation $t = T(\mathbf{x})$, then the three terms of the above equation have different orders of discontinuity, with the term in $\ddot{P}$ being the strongest, then the $\dot{P}$ term, and finally the P term. The strongest term is removed by requiring that

$$(\nabla T)^2 = \frac{1}{c^2}. \quad (2)$$

By requiring that the term in (1) in $\dot{P}$ also have a zero coefficient, we obtain an equation for the geometrical spreading factor A, namely

$$\nabla^2 T + 2\frac{\nabla A \cdot \nabla T}{A} = 0. \quad (3)$$

In general the final term, proportional to $P(t - T)$, will not be zero. The net result is that with T a solution of (2) and then A a solution of (3), the trial form $\phi = A(\mathbf{x})P(t - T(\mathbf{x})$ can be a useful approximate solution to $\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi$.

2.1.4 A GENERAL METHOD FOR SOLVING THE 3D WAVE EQUATION IN A HOMOGENEOUS MEDIUM

In Sections 2.1.1 and 2.1.2 we discussed the plane wave solution. Here, we shall find that such solutions can be used to build up more general solutions (for example, the waves from a point sources). The overall approach we shall develop, is related to the use of integral transforms to solve wave propagation problems.

Thus, a general method for solving

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi \quad \text{(2.10 again)}$$

is to try for a solution $\phi = \phi(\mathbf{x}, t)$ in the special form

$$\phi = \Phi(\mathbf{x})T(t) \quad (2.14)$$

in which the dependences on $\mathbf{x}$ and t come from different factors. This is the method of "separation of variables," and once we find a system of such solutions we can sum over the system and generate more general solutions, that do not factorize into separate dependences on space and time.

BOX **2.4**

On eigenvalues and eigenfunctions associated with ordinary differential equations

First, we note that the original partial differentiation with respect to t in (2.10) has become an ordinary differentiation in (2.15) and (2.16). Second, it is of interest that any of the functions $e^{i\omega t}$, $e^{-i\omega t}$, $\cos\omega t$, and $\sin\omega t$ can be thought of as eigenfunctions of the operator $\frac{d^2}{dt^2}$. That is, as functions of t they have the special property that double differentiation has the same effect as multiplying by a scalar $-(\omega^2)$:

$$\frac{d^2T}{dt^2} = -\omega^2 T. \tag{1}$$

So we can think of $-(\omega^2)$ as an eigenvalue by analogy with the algebra problem described in Box 1.2, namely

$$\mathbf{A}\cdot\mathbf{x} = \lambda\mathbf{x}. \tag{2}$$

This equation in general cannot have non-trivial solutions for the eigenvector, unless λ takes on special discrete values. A similar result often follows for the eigenvalues ω of (1). The discrete values of ω may come from a requirement that T is zero at two different values of t, or from similar spatial boundary conditions that give discrete values of the wavenumbers used in equations (2.21), hence constraining ω to discrete values by this same equation (2.21).

From (2.14) substituted into (2.10) we obtain $\frac{1}{c^2}\,\Phi(\mathbf{x})\,\ddot{T}(t) = \left(\nabla^2\Phi\right)\,T(t)$ and hence

$$\frac{\nabla^2\Phi}{\Phi} = \frac{1}{c^2}\frac{\ddot{T}}{T} \tag{2.15}$$

which has the form $f(t) = g(t)$. It follows that each side is a constant, since no variability with respect to $\mathbf{x}$ or t is allowed. [Note that we have previously introduced constants, and then allowed them to take different values and in this sense to become variables. We'll be doing this here also. When we say each side of (2.15) is a constant, we mean that this "constant" is independent of $\mathbf{x}$ and t.] If we say that each side of (2.15) is the constant $-k^2$, then

$$\frac{d^2T}{dT^2} + k^2c^2T = 0 \tag{2.16}$$

and if $\omega = kc$ then $T(t)$ must be a linear combination of $e^{i\omega t}$ and $e^{-i\omega t}$ (or of $\cos\omega t$ and $\sin\omega t$). We write this result as

$$T = e^{\pm i\omega t}. \tag{2.17}$$

The separated solution of (2.10) is now $\phi = \Phi(\mathbf{x},\omega)e^{\pm i\omega t}$, where

$$\nabla^2\Phi + \frac{\omega^2}{c^2}\Phi = 0. \tag{2.18}$$

If we were to sum over such solutions, for example in the form

$$\phi(\mathbf{x}, t) = \frac{1}{2\pi}\int \Phi(\mathbf{x}, \omega)e^{-i\omega t}\, d\omega \tag{2.19}$$

then we see there is a relation between the method of separation of variables, and the use of Fourier transforms. (In (2.19), ω ranges over all values so we have dropped the $\pm$ sign.) The solution $\phi(\mathbf{x}, t)$ here has been written as a linear combination (a summation, or in this case an integration), of separated solutions ΦT. In equation (2.19), the factor Φ in the integrand must satisfy equation (2.18) in order for ϕ to satisfy the scalar wave equation (2.10).

We can keep on going with the method of separation of variables — separating out the dependences on x_1, x_2, and x_3 via an assumption that

$$\Phi(\mathbf{x}) = X_1(x_1)X_2(x_2)X_3(x_3).$$

Then from (2.18) we find

$$\frac{X_1''}{X_1} + \frac{X_2''}{X_2} + \frac{X_3''}{X_3} + \frac{\omega^2}{c^2} = 0 \tag{2.20}$$

in which primes are used to denote spatial differentiation with respect to the appropriate argument (for example, $X_2' = \dfrac{dX_2}{dx_2}$). Because (2.20) has the form $f(x_1) + g(x_2) + h(x_3) + \dfrac{\omega^2}{c^2} = 0$, each of the four terms must be a constant. Say $\dfrac{X_1''}{X_1} = -k_1^2$ and $\dfrac{X_2''}{X_2} = -k_2^2$. We also have $\dfrac{X_3''}{X_3} = -k_3^2$ but now the choice of this last constant in terms of ω, k_1, and k_2, must satisfy

$$k_1^2 + x_2^2 + k_3^2 = k^2 = \frac{\omega^2}{c^2}. \tag{2.21}$$

This equation is a relation, equivalent to the original wave equation (2.10), between the four separation constants k_1, k_2, k_3, and ω. The fully separated solution has the form

$$\phi(\mathbf{x}, t) = e^{\pm ik_1x_1}e^{\pm ik_2x_2}e^{\pm ik_3x_3}e^{\pm i\omega t}. \tag{2.22}$$

If we wish, then without loss of generality we can drop the $\pm$ symbols as long as we recognize that the separation constants can take any values (subject to the contraint given by (2.21)). Or we can mix the constants in (2.22), some with plus signs and some with negative, so that for example

$$\phi(\mathbf{x}, t) = e^{i(k_1x_1+k_2x_2+k_3x_3-\omega t)} = e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{2.23}$$

is a 3D wave equation solution for any constants ω and $\mathbf{k} = (k_1, k_2, k_3)$ such that

$$\mathbf{k} \cdot \mathbf{k} = \frac{\omega^2}{c^2}. \qquad \text{(2.21 again)}$$

Note that (2.23) has the form of a plane wave $\phi = F\left(t - \frac{\mathbf{x} \cdot \mathbf{l}}{c}\right)$ if we make the identification $\frac{\mathbf{l}}{c} = \frac{\mathbf{k}}{\omega}$, or $\mathbf{l} = \frac{c}{\omega}\mathbf{k}$. The symbol $\mathbf{l}$ defined in this way is indeed a unit vector, because of (2.21). $\mathbf{k}$ is the *wavenumber vector*.

We can generalize (2.19) above, by noting that a linear combination of solutions (2.23) in the form

$$\phi(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \iiint X_3(k_1, k_2, x_3, \omega)\, e^{i(k_1 x_1 + k_2 x_2 - \omega t)}\, dk_1\, dk_2\, d\omega \qquad (2.24)$$

(where $\frac{\partial^2 X_3}{\partial x_3^2} + k_3^2 X_3 = 0$ and $k_3^2 = \frac{\omega^2}{c^2} - k_1^2 - k_2^2$) provides a solution to the three-dimensional wave equation (2.10). Since $X_3 \propto e^{\pm i k_3 x_3}$, the integrand of (2.24) is a plane wave.

Our overall conclusion of this Section, is that the method of separation of variables applied in a cartesian system of coordinates yields a plane wave solution. Each factor in (2.22) is an eigenfunction of a second order ordinary differential operator, as discussed in Box 2.4. And by linear combination of such separated solutions (in particular, by integration over plane waves, as in (2.24)) we can generate solutions to the 3D wave equation that in general are not separated solutions.

2.1.5 THE INTERACTION OF A PLANE WAVE WITH A PLANAR INTERFACE BETWEEN TWO DIFFERENT HOMOGENEOUS MEDIA: ACOUSTIC WAVES

First, we shall examine the case of an acoustic wave in a fluid, incident upon a planar interface. Later we shall find that plane waves generalize from those we have considered so far, to a type of wave in which amplitude decays exponentially with distance in a particular direction.

For a plane wave propagating in a fluid, rigidity is zero and the stress tensor is isotropic ($\tau_{ij} = -P\delta_{ij}$). So the wave equation $\rho \ddot{u}_i = \tau_{ij,j}$ becomes $\rho \ddot{\mathbf{u}} = -\nabla P$ and Hooke's law $\tau_{ij} = \lambda \nabla \cdot \mathbf{u} \delta_{ij} + 2\mu e_{ij}$ reduces to $-P = \lambda \nabla \cdot \mathbf{u}$. Since rigidity is zero there are no shear waves. The P-waves in a fluid that we are discussing here, are often called *acoustic waves*. They provide the method by which whales communicate in the oceans (where sound can travel for thousands of kilometers), and of course by which people communicate with sound waves in air. The study of acoustic waves has been extensively pursued, in the context of hunting for submarines. Oceanographers and seismologists use acoustic waves to study the ocean floor, and structures within the oceanic crust. Infrasound waves in the atmosphere are part of a developing technology to study winds at high altitude, and to monitor for meteorites, bolides, manmade explosions, and supersonic planes and space shuttles.

The wave equation (2.6) reduces in this case to $\rho \ddot{\mathbf{u}} = \lambda \nabla^2 \mathbf{u}$ since $\mu = 0$, but it is easier to quantify wave propagation in a fluid by analysing the pressure field P, because then we can work with a scalar rather than a vector as the dependent variable. It is easy to show (can

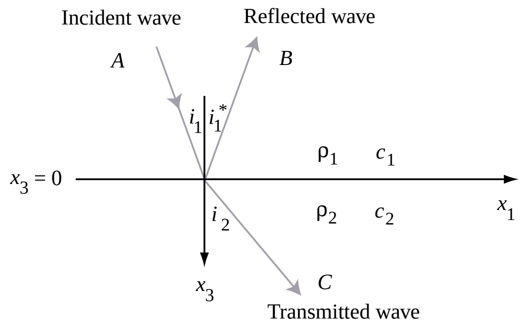


FIGURE **2.4**
Two homogeneous fluids, with densities and wave speeds $\rho_i, c_i (i = 1, 2)$, have a planar interface which is chosen as the plane $x_3 = 0$. An incident plane wave in the upper medium travels in the plane $x_2 = 0$ and is incident upon the interface with ray direction given by the angle i_1. The reflected wave (making an angle i_1^*) and transmitted wave (angle i_2) are also shown. Wave amplitudes are A, B, and C. The reflection and transmission coefficients are B/A and C/A respectively.

you do it?) that the scalar wave equation for pressure is

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = \nabla^2 P \qquad \text{where} \qquad c = \sqrt{\frac{\lambda}{\rho}}. \tag{2.25}$$

Each of the three plane waves shown in Figure 2.4 has a pressure field given by $P(\mathbf{x}, t) = F\left(t - \frac{\mathbf{x}\cdot\mathbf{l}}{c_i}\right)$ where $i = 1$ for the upper medium and $i = 2$ for the lower medium. The unit vector $\mathbf{l}$ is different for each of the three waves, being $\mathbf{l} = (\sin i_1, 0, \cos i_1)$ for the incident wave, $\mathbf{l} = (\sin i_1^*, 0, -\cos i_1^*)$ for the reflected wave, and $\mathbf{l} = (\sin i_2, 0, \cos i_2)$ for the transmitted wave. If we use the approach indicated in the previous Section, in which the dependencies on (x_1, x_2, x_3, t) are handled by separate factors as in $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ with $\mathbf{l} = \frac{c}{\omega}\mathbf{k}$, then we can take the incident wave as

$$P^{\text{inc}} = Ae^{i\omega\left(\frac{\sin i_1}{c_1}x_1 + \frac{\cos i_1}{c_1}x_3 - t\right)}. \tag{2.26}$$

The reflected wave is

$$P^{\text{refl}} = Be^{i\omega\left(\frac{\sin i_1^*}{c_1}x_1 - \frac{\cos i_1^*}{c_1}x_3 - t\right)}, \tag{2.27}$$

and the transmitted wave is

$$P^{\text{trans}} = Ce^{i\omega\left(\frac{\sin i_2}{c_2}x_1 + \frac{\cos i_2}{c_2}x_3 - t\right)}. \tag{2.28}$$

This is an example of linear wave propagation, so that we expect the amplitudes B and C of the scattered waves (reflected, transmitted) will increase in proportion to A, if the incident amplitude A is increased. The ratios B/A and C/A have yet to be determined along with the unknown angles i_1^* and i_2.

To solve for these unknowns we must use boundary conditions at the interface $x_3 = 0$. Across this surface, the vertical displacement u_3 must be continuous (otherwise a cavity would open up, or the two fluids would be driven to occupy the same volume). Also the total pressure field must be continuous (otherwise, a very thin disc of material with its upper face in medium 1 and its lower face in medium 2 would be subjected to a net force, and it would acquire an infinite acceleration). From $\rho\ddot{\mathbf{u}} = -\nabla P$, the first boundary condition translates into a requirement that $\frac{1}{\rho}\frac{\partial P}{\partial x_3}$ has the same value just above the interface (in medium 1), as its value just below the interface (in medium 2). That is,

$$\frac{i\omega}{\rho_1}\left[A\frac{\cos i_1}{c_1}e^{i\omega\left(\frac{\sin i_1}{c_1}x_1-t\right)} - B\frac{\cos i_1^*}{c_1}e^{i\omega\left(\frac{\sin i_1^*}{c_1}x_1-t\right)}\right] = \frac{i\omega}{\rho_2}C\frac{\cos i_2}{c_2}e^{i\omega\left(\frac{\sin i_2}{c_2}x_1-t\right)}. \quad (2.29)$$

The second boundary condition (continuity of pressure) is simpler, namely

$$Ae^{i\omega\left(\frac{\sin i_1}{c_1}x_1-t\right)} + Be^{i\omega\left(\frac{\sin i_1^*}{c_1}x_1-t\right)} = Ce^{i\omega\left(\frac{\sin i_2}{c_2}x_1-t\right)}. \quad (2.30)$$

Both (2.29) and (2.30) apply for all values of x_1 and t. It follows that

$$\frac{\sin i_1}{c_1} = \frac{\sin i_1^*}{c_1} = \frac{\sin i_2}{c_2}. \quad (2.31)$$

This result, known as Snell's law, essentially says that the component of wavenumber $\mathbf{k}$ taken along the interface, is the same for each of the scattered waves (two, in this case), as it is for the incident wave. The angle of reflection, i_1^* equals the incident angle i_1, and the transmission angle i_2 is simply related to i_1. In optics where there is a similar result, it is more common to work with refractive index than wave speed. Since refractive index n is inversely proportional to speed c, Snell's law has the form $n_1 \sin i_1 = n_2 \sin i_2$ in optics.

Because of Snell's law, all the exponentials in equations (2.29) and (2.30) are the same and can be cancelled out leaving the following two equations for the ratio B/A and C/A:

$$\frac{\cos i_1}{\rho_1 c_1}(A - B) = \frac{\cos i_2}{\rho_2 c_2}C \qquad \text{and} \qquad A + B = C,$$

which are easily solved to give

$$\frac{B}{A} = \frac{\dfrac{\cos i_1}{\rho_1 c_1} - \dfrac{\cos i_2}{\rho_2 c_2}}{\dfrac{\cos i_1}{\rho_1 c_1} + \dfrac{\cos i_2}{\rho_2 c_2}}, \qquad \text{and} \qquad \frac{C}{A} = \frac{2\dfrac{\cos i_1}{\rho_1 c_1}}{\dfrac{\cos i_1}{\rho_1 c_1} + \dfrac{\cos i_2}{\rho_2 c_2}}. \quad (2.32)$$

The product ρc which appears here repeatedly (evaluated for the upper and lower media) is the *impedance* of a fluid. Impedance is essentially the ratio of pressure to particle velocity. Impedance is high, if high pressure leads only to small particle velocity. Impedance is low, if the particle motion is large even at low pressure.

Note that the reflection coefficient B/A and the transmission coefficient C/A are both real, if angles i_1 and i_2 exist in the range from 0° to 90°. If $i_1 = 0$ then $i_2 = 0$, and the coefficients reduce to

$$\frac{B}{A} = \frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1} \qquad \text{and} \qquad \frac{C}{A} = \frac{\rho_2 c_2}{\rho_2 c_2 + \rho_1 c_1}.$$

For a free surface, where there is just one fluid and the pressure is zero at the surface of a half space, an incident pressure wave is reflected back with $B/A = -1$, or $B = -A$. (Take $\rho_2 = 0$ in the above equation to see this — the result is true also for general angles of incidence, as can be shown from (2.32).) So the reflected wave of pressure has exactly the same amplitude (but the opposite sign) of the incident pressure. This is why the pressure from the two waves, taken together, cancels out, to give zero pressure at the free surface itself.

2.1.6 EVANESCENT WAVES

Having obtained algebraic formulas for the reflection and transmission coefficients of an acoustic wave in (2.32), we ask what will happen if the incident angle i_1 is increased to an angle large enough to prevent a simple solution i_2 being given by Snell's law, $\dfrac{\sin i_2}{c_2} = \dfrac{\sin i_1}{c_1}$. This situation can easily arise if $c_2 > c_1$, for which Snell's law usually implies that $i_2 > i_1$, as shown in Figure 2.4. The essence of this latter Figure is repeated in Figure 2.5. The transmitted angle i_2 is shown in this case as being less than 90°, but if $\dfrac{\sin i_1}{c_1} > \dfrac{1}{c_2}$ then we can no longer find a real angle i_2 because then $\sin i_2 > 1$. In such cases, there exist waves in the lower medium that exponentially grow or exponentially decay with depth.

In cases such that $p > \frac{1}{c_2}$, the transmitted wave of (2.28) becomes

$$Ce^{i\omega(px_1 - t)}\, e^{i\omega\sqrt{\frac{1}{c_2^2} - p^2}\, x_3} = Ce^{-\omega\sqrt{p^2 - \frac{1}{c_2^2}}\, x_3}\, e^{i\omega(px_1 - t)}. \tag{2.33}$$

When interpreting the first square root, on the left-hand side of (2.33), we have made the choice $\sqrt{\frac{1}{c_2^2} - p^2} = +i\sqrt{p^2 - \frac{1}{c_2^2}}$ because this gives a negative exponential on the right-hand side of (2.33), and then the wave decays with depth below the interface at $x_3 = 0$. This is an example of an evanescent wave. If we had made the other choice, $\sqrt{\frac{1}{c_2^2} - p^2} = -i\sqrt{p^2 - \frac{1}{c_2^2}}$, then we would obtain an exponentially growing wave. Although both waves are solutions of the wave equation, in practice we are usually more interested in the exponentially decaying solutions because they satisfy the condition that no radiation be transmitted to great depths.

It is a convenience if we assign a new label, p, to the value of $\dfrac{\sin i_1}{c_1} = \dfrac{\sin i_2}{c_2}$ used in Snell's law. This quantity, sometimes called the *ray parameter*, is the same for all three rays shown in Figure 2.5. All of the algebraic manipulations we did in going from (2.26) to (2.32) are still valid in the case $p > \dfrac{1}{c_2}$ and $\sin i_2 > 1$, provided we interpret $\dfrac{\sin i_2}{c_2}$ as p, and $\dfrac{\cos i_2}{c_2}$ as $\sqrt{\dfrac{1}{c_2^2} - p^2} = i\sqrt{p^2 - \dfrac{1}{c_2^2}}$.

Proceeding further with the choice of square root made in (2.33), we can go to (2.32) and find for $p > \dfrac{1}{c_2}$ that now

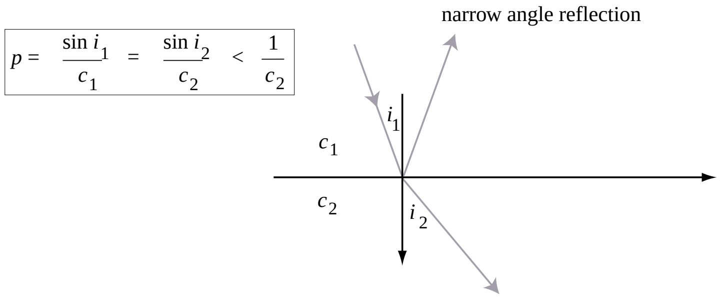


FIGURE **2.5**
Narrow angle incidence upon a planar interface: $p < \dfrac{1}{c_2}$. A transmitted wave exists, propagating away from the interface.

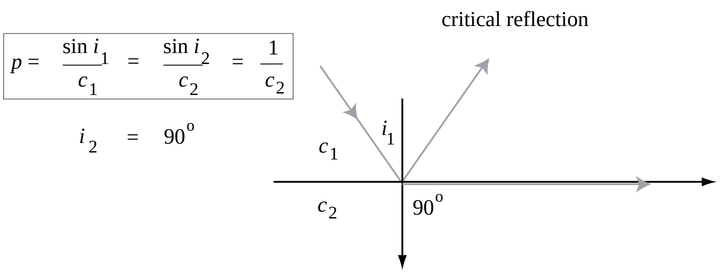


FIGURE **2.6**
Critical incidence upon a planar interface: $p = \dfrac{1}{c_2}$. The transmitted wave in the lower (faster) medium, propagates parallel to the interface.

$$\frac{B}{A} = \frac{\dfrac{\cos i_1}{\rho_1 c_1} - \dfrac{i}{\rho_2}\sqrt{p^2 - \frac{1}{c_2^2}}}{\dfrac{\cos i_1}{\rho_1 c_1} + \dfrac{i}{\rho_2}\sqrt{p^2 - \frac{1}{c_2^2}}}. \tag{2.34}$$

Because this expression has the form

$$\frac{B}{A} = \frac{a - ib}{a + ib} \quad \text{with real values of } a \text{ and } b$$

it follows that

$$\left|\frac{B}{A}\right| = 1.$$

A similar example of wide-angle incidence (Figure 2.7 with $c_1 =$ the speed of light in glass, and $c_2 =$ speed of light in air) in optics is called *total internal reflection*, and there is little interest in the evanescent wave below the boundary because ω is so high that the

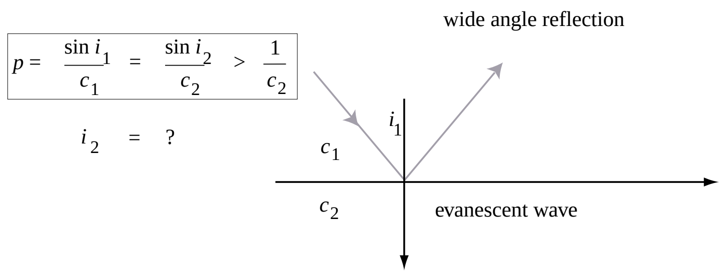


FIGURE **2.7**
Wide angle incidence upon a planar interface: $p > \dfrac{1}{c_2}$. The transmitted wave has an amplitude that decays exponentially with distance from the interface.

exponentially decaying transmitted wave (2.33) has little importance. For example, this is the situation in the use of total internal reflection in a pair of binoculars ($i_1 = 45^\circ$, medium 1 is glass, and refractive index of glass = speed of light in a vacuum ÷ speed of light in glass $\sim$ 1.5, so that $p > \dfrac{1}{c_2}$). But in many other fields, including geophysics, the evanescent waves are very important. They represent a class of waves, sometimes called *inhomogeneous* waves, that satisfy the wave equation with horizontal oscillation and vertical decay of amplitude. In terms of the "unit vector" $\mathbf{l} = (l_1, 0, l_3)$, we still have $l_1^2 + l_3^2 = 1$ and we still have $\mathbf{l} = (\sin i_2, 0, \pm \cos i_2)$. But now $l_1 = \sin i_2 > 1$ and $l_3 = \pm \cos i_2$ is imaginary.

2.1.7 THE INTERACTION OF A PLANE WAVE WITH THE "FREE SURFACE" OF AN ELASTIC HALF-SPACE

Here we shall consider plane P-waves and plane SV-waves, incident from below upon the planar free surface of an elastic solid as shown in Figure 2.8.

By "free surface", we mean a surface that has no traction. Taking the surface to be horizontal, and the x_3-axis as the depth direction, this means that the stress tensor components τ_{31}, τ_{32}, and τ_{33} are all zero on $x_3 = 0$.

In Figure 2.8, if we assume the incident P-wave has a displacement with unit amplitude and frequency ω, its displacement is given by

$$\text{incident } P\text{-wave, of unit displacement} = (\sin i, 0, -\cos i)e^{i\omega\left(px_1 - \frac{\cos i}{\alpha}\, x_3 - t\right)}. \quad (2.35)$$

The only significant difference between this expression and (2.26), is that (2.35) is a vector instead of a scalar. (The first term on the right-hand side of (2.35) is a unit vector in the longitudinal direction corresponding to the incident P-wave wavefront.)

In order to determine what waves are reflected from the free surface at $x_3 = 0$, and with what amplitude, we need to take account of the boundary conditions. For a free surface, there is no constraint on displacement. But $x_3 = 0$ is a traction-free surface, so $\tau_{31} = \tau_{32} = \tau_{33} = 0$ on $x_3 = 0$. If we allow for a P-wave reflection, its form will be

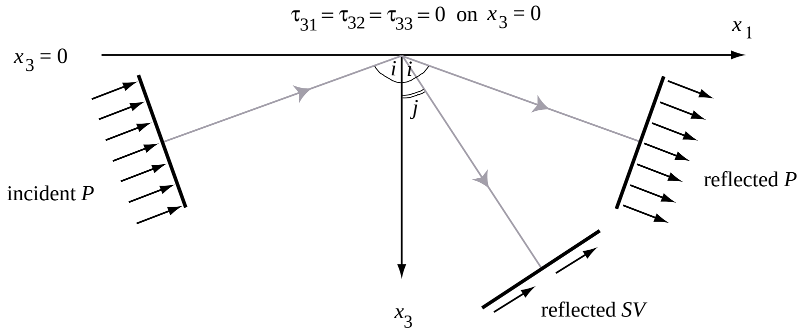


FIGURE **2.8**
A plane P-wave is shown, incident upwards upon the free surface of an elastic half-space. Taking this surface as the plane $x_3 = 0$, with the x_3-axis as the depth direction, and the x_1 axis as the horizontal direction containing the horizontal component of the P-wave motion, it follows that the key boundary conditions are $\tau_{31} = 0$ and $\tau_{33} = 0$ on $x_3 = 0$. (While it is also true that $\tau_{32} = 0$ on $x_3 = 0$, this latter condition is trivially satisfied because τ_{32} is not excited by the incident wave, and remains zero everywhere.) Rays are shown in grey, the position of wavefronts is shown as heavy black lines perpendicular to the rays, and short lines with small arrows indicate the directions of particle motion (longitudinal for the two P-waves, and transverse for the S-wave). Because the transverse component of S in this case lies in a vertical plane, the reflected S-wave is polarized as SV, using terminology introduced at the end of Section 2.1.1.

$$\text{reflected } P\text{-wave} = \acute{P}\grave{P}\ (\sin i, 0, \cos i)e^{i\omega\left(px_1 + \frac{\cos i}{\alpha}\ x_3 - t\right)}. \tag{2.36}$$

Note here that the phase of the wave increases in the longitudinal direction given by unit vector $\mathbf{l} = (\sin i, 0, +\cos i)$, which is also the direction of particle motion (because this is a P-wave).

For both the incident wave (2.35) and the reflected wave (2.36), it is easy to use $\tau_{ij} = \lambda\nabla\delta_{ij} + \mu(u_{i,j} + u_{j,i})$ to conclude that neither wave perturbs the τ_{32} component of stress, and both waves perturb the τ_{31} and τ_{33} components. It follows that if the only reflected wave is the P-wave described in (2.36), we can satisfy the requirement that $\tau_{32} = 0$ on $x_3 = 0$. (In fact, (2.35) and (2.36) have $\tau_{32} = 0$ everywhere.) But we cannot satisfy both $\tau_{31} = 0$ and $\tau_{33} = 0$ on $x_3 = 0$. To satisfy both these scalar boundary conditions, we need to allow for another reflected wave, namely the reflected S-wave shown in Figure 2.8. Its displacement is given by

$$\text{reflected } SV\text{-wave} = \acute{P}\grave{S}\ (\cos j, 0, -\sin j)e^{i\omega\left(px_1 + \frac{\cos j}{\beta}\ x_3 - t\right)}. \tag{2.37}$$

The terms in the exponential here are chosen to make this wave travel downwards at an angle j determined by $\dfrac{\cos j}{\beta} = \dfrac{\cos i}{\alpha} = p$ (an extension of Snell's law to cover different wave types). And, the displacement here has a vector direction which is transverse. Because the particle motion in the incident and reflected P-waves is confined to the x_1–x_3 plane, S-wave motion can be expected to be confined to this same plane. So, the S-wave has SV polarization (see Section 2.1.1), with no displacement component in the x_2 direction in the present problem, and no excitation of the τ_{32} component of stress.

Equations (2.36) and (2.37) introduce a notation for the reflection coefficients of the reflected waves, namely $\acute{P}\grave{P}$ and $\acute{P}\grave{S}$, that in miniature indicates which wave is incident, and which is reflected. Aki and Richards (1980, 2002) use this notation to analyse in detail all 16 possibilities if P-waves are incident from above or below, upon the planar interface between two different solid elastic half-spaces, and each incident wave generates upgoing and downgoing P- and S-waves in each half-space.

To apply the two non-trivial boundary conditions, which are given in terms of stress components, we need to evaluate the relevant stresses in terms of displacements (2.36) and (2.37) using the general stress–strain relation (1.37) in an isotropic solid. Thus, since $\tau_{31} = \mu\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)$, some algebra turns the boundary condition $\tau_{31} = 0$ on $x_3 = 0$ into the first relation between $\acute{P}\grave{P}$ and $\acute{P}\grave{S}$. It is

$$2\beta p\ \cos i\ \acute{P}\grave{P} + (1 - 2\beta^2 p^2)\ \acute{P}\grave{S} = 2\beta p\ \cos i. \tag{2.38}$$

And since $\tau_{33} = \lambda\nabla\cdot\mathbf{u} + 2\mu\frac{\partial u_3}{\partial x_3}$, the boundary condition $\tau_{33} = 0$ on $x_3 = 0$ provides the second relation (after more algebra) as

$$\alpha(1 - 2\beta^2 p^2)\ \acute{P}\grave{P} - 2\beta^2 p\cos j\ \acute{P}\grave{S} = -\alpha(1 - 2\beta^2 p^2). \tag{2.39}$$

At last, in (2.38) and (2.39) we have two equations for the two unknown reflection coefficients. The solutions are

$$\acute{P}\grave{P} = \frac{-\left(\frac{1}{\beta^2} - 2p^2\right)^2 + 4p^2\frac{\cos i}{\alpha}\frac{\cos j}{\beta}}{+\left(\frac{1}{\beta^2} - 2p^2\right)^2 + 4p^2\frac{\cos i}{\alpha}\frac{\cos j}{\beta}}, \tag{2.40}$$

and

$$\acute{P}\grave{S} = \frac{4\frac{\alpha}{\beta}p\frac{\cos i}{\alpha}\left(\frac{1}{\beta^2} - 2p^2\right)}{\left(\frac{1}{\beta^2} - 2p^2\right)^2 + 4p^2\frac{\cos i}{\alpha}\frac{\cos j}{\beta}}. \tag{2.41}$$

2.1.8 RAYLEIGH WAVES

In Section 2.1.6 we showed that evanescent waves, or inhomogeneous waves, can exist as solutions of the acoustic wave equation. These acoustic P-waves propagate horizontally (the x_1 direction) with a phase factor given by $e^{i\omega(px_1-t)}$, and they decay exponentially with depth x_3. Their horizontal speed is $1/p$.

More than a hundred years ago, Lord Rayleigh showed that it is possible to have a pair of evanescent waves, one of P-wave type, the other an SV-wave, which when added together can satisfy the free surface boundary conditions. But this coupled solution, a superposition of P and SV, can occur only when the constant p takes a special value.

To analyse this possibility, we can write (2.36) and (2.37) in the notation that is more appropriate when $\frac{1}{\alpha} < \frac{1}{\beta} < p$. (Another way to state these inequalities, is to say that p is the ray parameter for a wave that travels horizontally with a speed that is slower than a horizontally-propagating P-wave, and slower also than a horizontally-propagating S-wave.) Since the angles i and j of Section 2.1.7 and Figure 2.8 cannot be given real values, we make the following interpretations:

$$(\sin i, 0, \cos i)e^{i\omega\left(px_1+\frac{\cos i}{\alpha}x_3-t\right)} = (\alpha p, 0, i\sqrt{\alpha^2p^2-1})\, e^{-\omega\sqrt{p^2-\frac{1}{\alpha^2}}\,x_3}\, e^{i\omega(px_1-t)},$$

and (2.42)

$$(\cos j, 0, -\sin j)e^{i\omega\left(px_1+\frac{\cos j}{\beta}x_3-t\right)} = (i\sqrt{\beta^2p^2-1}, 0, -\beta p)\, e^{-\omega\sqrt{p^2-\frac{1}{\beta^2}}\,x_3}\, e^{i\omega(px_1-t)}.$$

Because the x_3 component of each of these inhomogeneous waves has a phase that is greater than the x_1 component by 90° (see the right-hand sides of both the first and second of equations (2.42)), the particle motion of each of the two waves is elliptical (see Problem 2.6).

Suppose now that we form a linear combination of the P- and SV-waves given in (2.42), and see if the combined waves can be made to satisfy both of the non-trivial boundary conditions $\tau_{31} = \tau_{33} = 0$ of a free surface. If we take $\grave{P}$ times the first of (2.42) and add it to $\grave{S}$ times the second of (2.42), then

$$2p\alpha\beta i\sqrt{p^2-\frac{1}{\alpha^2}}\;\grave{P} + (1-2\beta^2p^2)\;\grave{S} = 0 \quad (\text{from } \tau_{31} = 0 \text{ on } x_3 = 0), \qquad (2.43)$$

and

$$(1-2\beta^2p^2)\;\grave{P} - \frac{2\beta^3pi}{\alpha}\sqrt{p^2-\frac{1}{\beta^2}}\;\grave{S} = 0 \quad (\text{from } \tau_{33} = 0 \text{ on } x_3 = 0). \qquad (2.44)$$

(We choose the notation $\grave{P}$ and $\grave{S}$ for the coefficients of the two waves, because these constants symbolize the amounts of the downgoing P- and SV-waves that we are combining, in the case that p is small and it is natural to work with the left-hand sides of the two equations in (2.43). When p is large, so that the right-hand sides of equations (2.43) are more appropriate and the waves exponentially decay with depth, then $\grave{P}$ and $\grave{S}V$ determine how much of each decaying solution is present in the combination.)

In general it is not possible to satisfy both these equations at once, unless $\grave{P} = \grave{S} = 0$. (Essentially, they are two equations for the ratio between $\grave{P}$ and $\grave{S}$.) But they *can* both be satisfied with non-trivial values of $\grave{P}$ and $\grave{S}$ if the determinant of coefficients vanishes. This requires that $\mathsf{R}(p) = 0$, where

$$\begin{aligned}\mathsf{R}(p) &\equiv \left(\frac{1}{\beta^2}-2p^2\right)^2 - 4p^2\sqrt{p^2-\frac{1}{\alpha^2}}\,\sqrt{p^2-\frac{1}{\beta^2}}\\ &= \left(\frac{1}{\beta^2}-2p^2\right)^2 + 4p^2\frac{\cos \mathrm{i}}{\alpha}\frac{\cos \mathrm{j}}{\beta}.\end{aligned} \qquad (2.45)$$

This function of p^2 has just one zero (for positive imaginary "cosines"), which is real and positive. Since the corresponding positive value of p is slightly (4–14%) greater than $1/\beta$ for all elastic solids, it is indeed possible for a coupled pair of inhomogeneous waves, P and SV, to propagate along the free surface of a half-space. Such a surface wave is named for Rayleigh, who described its main properties in 1885. (Lord Rayleigh had enormous scientific accomplishments in the application of mathematical methods to learn for the first time about fundamental properties of acoustic waves, elastic waves, and non-linear motions including convection. He was awarded the Nobel Prize in physics for his discovery of argon in the Earth's atmosphere.)

When an earthquake or an underground explosion occurs, the seismic body waves such as P and S spread throughout the three-dimensional volume of the Earth's interior. In the simplest case of a homogeneous medium, the wavefronts of these body waves are expanding spheres, and the amplitude decreases with distance r like $\frac{1}{r}$ (see, for example, (2.13)). But the Rayleigh wave spreads only over the Earth's surface, expanding like a circle rather than a sphere, and therefore does not attenuate so rapidly. In fact, surface-wave amplitudes attenuate with distance like $\frac{1}{\sqrt{r}}$. This means that the ratio of surface-wave amplitude to body-wave amplitude increases like $\sqrt{r}$ with distance r from the seismic source, so that the surface waves become progressively stronger and stronger relative to body waves.

Suggestions for Further Reading

Menke, William, and Dallas Abbott. *Geophysical Theory*, New York: Columbia University Press, 1990 (pp 253–260 for properties of plane waves in a fluid, and pp 326–330 for plane waves in an elastic solid).

Aki, Keiiti, and Paul G. Richards. *Quantitative Seismology*, second edition, Sausalito, California: University Science Books, 2002 (Chapter 5, for plane waves and their interaction with a plane boundary).

Problems

2.1 Show that when the general plane wave trial solution (2.9) is substituted into (2.6), the vector wave equation in the absence of body forces becomes

$$\rho\ddot{\mathbf{u}} = \frac{\lambda + 2\mu}{c^2}(\ddot{\mathbf{u}} \cdot \mathbf{l})\mathbf{l} - \frac{\mu}{c^2}\mathbf{l} \times (\mathbf{l} \times \ddot{\mathbf{u}}).$$

Taking the scalar product and the vector product of this result with $\mathbf{l}$ (that is, $\mathbf{l} \cdot \ldots$ and $\mathbf{l} \times \ldots$), show that

$$\left(\rho - \frac{\lambda + 2\mu}{c^2}\right)\ddot{\mathbf{u}} \cdot \mathbf{l} = 0$$

and

$$\left(\rho - \frac{\mu}{c^2}\right)\ddot{\mathbf{u}} \times \mathbf{l} = \mathbf{0}.$$

[The last three equations are valid for any plane wave in the form (2.9). So, up to this point, don't make any assumptions about whether the wave is longitudinal or transverse in the derivations.]

From the last two equations show that *either* $c^2 = \dfrac{\lambda + 2\mu}{\rho}$ and $\ddot{\mathbf{u}} \times \mathbf{l} = \mathbf{0}$, *or* $c^2 = \dfrac{\mu}{\rho}$ and $\ddot{\mathbf{u}} \cdot \mathbf{l} = 0$. Show finally that the plane wave travels with speed $c = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$ and has motion parallel to $\mathbf{l}$ and is therefore longitudinal (this is the plane P-wave); or the plane wave travels with speed $c = \sqrt{\dfrac{\mu}{\rho}}$ and has motion perpendicular to $\mathbf{l}$ so that it is transverse (this is the plane S-wave). There are no other types of elastic plane wave in an isotropic homogenous medium.

2.2 If an elastic displacement satisfies the vector equation (2.6), and if this displacement and the body force are represented by potentials so that $\mathbf{u} = \nabla\phi + \nabla\boldsymbol{\psi}$ (with $\nabla \cdot \boldsymbol{\psi} = 0$) and $\mathbf{f} = \nabla\Phi + \nabla\boldsymbol{\Psi}$ (with $\nabla \cdot \boldsymbol{\Psi} = 0$), show that (2.6) becomes a third-order partial differential equation. Show that separate fourth-order equations for ϕ and $\boldsymbol{\psi}$ can be written in the form

$$\nabla^2[\rho\ddot{\phi} - (\lambda + 2\mu)\nabla^2\phi - \Phi] = 0$$

and

$$\nabla^2[\rho\ddot{\boldsymbol{\psi}} - \mu\nabla^2\boldsymbol{\psi} - \boldsymbol{\Psi}] = \mathbf{0}.$$

[To generate a solution of (2.6), note that it is sufficient to require ϕ and $\boldsymbol{\psi}$ to satisfy the simpler second-order equations $\rho\ddot{\phi} - (\lambda + 2\mu)\nabla^2\phi = \Phi$ and $\rho\ddot{\boldsymbol{\psi}} - \mu\nabla^2\boldsymbol{\psi} = \boldsymbol{\Psi}$, because such solutions also satisfy the third-order differential equation mentioned above. But is it true that all possible displacement solutions $\mathbf{u}$ to (2.6) can be generated by potentials ϕ and $\boldsymbol{\psi}$ that satisfy these second-order wave equations? Fortunately the answer here is "yes," though a proof was not given for more than 100 years after it was assumed to be true. It follows that only second-order equations for ϕ and $\boldsymbol{\psi}$ are needed, in order to generate all possible solutions $\mathbf{u}$.]

Show from $\mathbf{u} = \nabla\phi + \nabla\boldsymbol{\psi}$ (with $\nabla \cdot \boldsymbol{\psi} = 0$) that the potential ϕ generates a displacement which is irrotational (has zero curl); and that the potential $\boldsymbol{\psi}$ generates a displacement which is divergence-free (no volume change). [Thus the P-wave, in addition to having the characteristic speed $c = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$ and being longitudinal, is also irrotational but carries a change in volume; and the S-wave, in addition to having the characteristic speed $c = \sqrt{\dfrac{\mu}{\rho}}$ and being transverse, is also divergence-free (sometimes called equivoluminal) but carries a change in particle rotation. P-wave motion is sometimes called compressional, entailing dilatation or rarefaction as well as compression. An S-wave entails shearing motion.]

2.3 For the problem of a plane wave incident upon the interface between two fluids

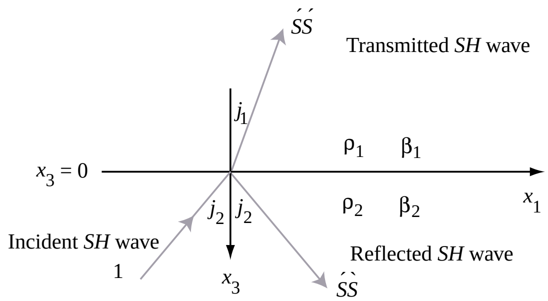


FIGURE **2.9**
The three *S* waves shown here have horizontal particle motions (thus giving *SH* motion).

(see Section 2.1.5 and Figure 2.4), show that in general the horizontal component of displacement is discontinuous across the interface.

For the fluids shown in Figure 2.4, $\rho_2 > \rho_1$ for stability. If also $c_2 > c_1$, then show that the bulk modulus (defined in Problem 1.6) of the lower fluid is greater than that of the upper one.

What types of physical phenomena might result from the discontinuity in horizontal displacement?

2.4 Prove the statement made following (2.32), that "Impedance is essentially the ratio of pressure to particle velocity."

2.5 Suppose that an SH wave is incident from below, upon the surface $x_3 = 0$ as shown in Figure 2.9.

a) What are the two boundary conditions needed, to determine the two coefficients $\acute{S}\acute{S}$ (transmission) and $\acute{S}\grave{S}$ (reflection)?
b) Show that the displacement

$$\mathbf{u} = (0, u_2^{\text{inc}}, 0) \quad \text{where} \quad u_2^{\text{inc}} = e^{i\omega\left(\frac{\sin j_2}{\beta_2}x_1 - \frac{\cos j_2}{\beta_2}x_3 - t\right)} = e^{i\omega\left(px_1 - \frac{\cos j_2}{\beta_2}x_3 - t\right)}$$

is a satisfactory form for the incident wave (i.e., show that **u** given by these formulas is a plane SH wave, propagating in the correct direction and with unit amplitude).
c) Write down the corresponding formulas for the transmitted u_2 component, which can be called u_2^{trans}, and the reflected component, u_2^{refl}.
d) Show that the two coefficients in this problem are given by

$$\acute{S}\acute{S} = \frac{2\rho_2\beta_2 \cos j_2}{\rho_1\beta_1 \cos j_1 + \rho_2\beta_2 \cos j_2} \quad \text{and} \quad \acute{S}\grave{S} = -\frac{\rho_1\beta_1 \cos j_1 - \rho_2\beta_2 \cos j_2}{\rho_1\beta_1 \cos j_1 + \rho_2\beta_2 \cos j_2}.$$

e) If $x_3 = 0$ is a free surface, on which $\tau_{32} = 0$, then there is no transmitted wave. Show in this case that the particle motion of the free surface itself is double the particle motion in the incident wave.

2.6 Show from the left-hand sides of (2.42) that for each of these two waves, P and SV, the particle motion is linear when $p < \frac{1}{\alpha} < \frac{1}{\beta}$. [By "linear", we mean the particles move in a line, with their x_1 and x_3 components either in phase, or exactly out of phase (by 180°). So, for a homogeneous P-wave or SV-wave, the particles move in straight lines — longitudinal or transverse. The whole point of this problem, given below, is to make the point that particle motion for inhomogeneous waves is *not* linear.]

The right-hand sides of equations (2.42) give the form of both P- and SV-waves when their horizontal speed $\frac{1}{p}$ is so slow that the waves decay exponentially with depth x_3. Conventionally we express the wave by taking the real part of these equations. Show for the inhomogeneous P-wave described by the right-hand side of (2.42) that the particle motions $(u_1, 0, u_3)$ satisfy

$$\frac{u_1^2}{\alpha^2 p^2} + \frac{u_3^2}{\alpha^2 p^2 - 1} = e^{-2\omega\sqrt{p^2 - \frac{1}{\alpha^2}}\, x_3}$$

and hence that the particle motion in an inhomogeneous P-wave is elliptical.

What is the corresponding result for the inhomogeneous SV-wave, derived from the second of (2.42)?

2.7 If $\lambda = \mu$, show that the Rayleigh wave function $\mathsf{R}(p)$ given in (2.45) is zero if $\frac{1}{p^2\beta^2} = 2$ or $2 \pm \frac{2}{\sqrt{3}}$.

Which of these possibilities can provide a coupled inhomogeneous P-wave and SV-wave?