CHAPTER 5

Complex Functions of a Complex Variable

This chapter briefly reviews the special properties of complex-valued functions of (x, y) where x and y are real, and the dependence on this pair of independent variables is only via the combination $z = x + iy$.

If x and y are real variables, and $z = x + iy$, we say that the complex function $f(z)$ is an analytic function of z if

(i) the derivative $\dfrac{df}{dz}$ exists, and
(ii) this derivative is independent of the orientation of δz in the limit as $\delta z \to 0$.

If the real part of f is u, and the imaginary part of f is v, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{wherever } f(z) \text{ is analytic.} \tag{5.1}$$

These are known as the Cauchy-Riemann relations, and they are easy to prove by taking the limit

$$\frac{\delta f}{\delta z} \to \frac{df}{dz}$$

in two different ways. First, let $\delta z = \delta x$, and second, let $\delta z = i\delta y$. In the first case, we have

$$\begin{aligned}\frac{df}{dz} &= \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0} \frac{u(x + \delta x, y) + iv(x + \delta x, y) - u(x, y) - iv(x, y)}{\delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.\end{aligned}$$

In the second case, we have

$$\begin{aligned}\frac{df}{dz} &= \lim_{\delta y \to 0} \frac{u(x, y + i\ \delta y) + iv(x, y + i\ \delta y) - u(x, y) - iv(v, y)}{i\ \delta y} \\ &= \frac{\partial u}{i\ \partial y} + \frac{i\ \partial v}{i\ \partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.\end{aligned}$$

BOX **5.1**

A practical application of complex function theory

When integral transforms are used to solve differential equations, it is common to find that an algebraic expression can be obtained for the transformed solution. The untransformed solution can then obtained as an inverse transform, for example as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)\, e^{ikx} dk$$

where $F(k)$ is a known algebraic expression.

In practice, this integral can often be evaluated by treating the independent variable of integration (k, in our example) as a complex-valued quantity, and the integrand as a complex function of a complex variable. By changing the integration so that it follows a complex path (in the complex k–plane, in our example, instead of going along the real k–axis), the result of the integration can be written in many different forms, some of which may easily allow the integral to be evaluated. An example is given at the end of this chapter, where $F(k) = \dfrac{1}{k^2 + a^2}$.

Since the formulas for the derivative of f must give the same result in these two cases,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

and by taking the real and imaginary parts of this equation we see that the Cauchy-Riemann relations (5.1) must be true.

Note that u and v are both real functions of the real variables x and y. What makes the subject of "complex functions of a complex variable" so special, and quite different from simply studying $f(x, y) = u(x, y) + iv(x, y)$, is that we are treating x and y in the combination $z = x + iy$. In this case we can apply the Cauchy-Riemann relations for any values of z for which f is analytic.

As an example of the special properties of f at values of z where f is analytic, note that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0.$$

Similarly $\nabla^2 v = 0$, and hence $\nabla^2 f = 0$. Thus, analytic functions are special functions of (x, y) in that they satisfy the two-dimensional Laplace equation.

Analytic functions have numerous other properties which lead to many practical applications. The most important properties stem from Cauchy's Theorem, proved in the next section. The applications usually entail the use of functions that are not analytic everywhere — the most important examples arising from singularities of $f(z)$, discussed in a later section.

5.1 Cauchy's Theorem

If f is analytic as a function of z, for values of z both on and inside a closed curve C in the complex z plane, then

$$\oint_C f\, dz = 0. \tag{5.2}$$

(We use the notation $\oint_C$, rather than $\int_C$, when we wish to indicate that C is a closed curve and the path of integration is taken around the whole circuit in the positive direction, that is, anticlockwise.[1])

This major result, which was obtained by Auguste Cauchy in the 1820s, can be proved by an application of Gauss's divergence theorem

$$\int_V \nabla \cdot \mathbf{A}\, dV = \int_S \mathbf{A} \cdot \mathbf{dS}. \tag{5.3}$$

Note that (5.3) applies to real vectors and surfaces and volumes in three dimensions. But if $\mathbf{A}$ depends only on the real variables (x, y) then we can consider a prism whose cross-section is the curve C of (5.2), allowing us to obtain a two-dimensional version of Gauss's theorem in the form

$$\int_S \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right) dS = \oint_C (A_x, A_y).(dy, -dx). \tag{5.4}$$

Here, we have changed notation from (5.3) and are now using S as the interior of the curve C. As shown in Figure 5.1, the vector $(dy, -dx)$ is in the outward direction from C, required in (5.4) as the interpretation of the right-hand side of (5.3).

We can use (5.4) to prove (5.2) by first writing (5.2) as

$$\oint_C f\, dz = \oint_C (u + iv)(dx + i dy) = \oint_C [(u\, dx - v\, dy) + i(v\, dx + u\, dy)]. \tag{5.5}$$

If we then define the two-dimensional real vector $\mathbf{A}$ as (v, u), (5.4) gives

$$\int_S \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dS = \oint_C (v, u)(dy, -dx) = -\oint_C (u\, dx - v\, dy). \tag{5.6}$$

But since $f(z)$ is analytic everywhere in S (the interior of C), one of the Cauchy-Riemann relations (5.1) tells us that the left-hand side of (5.6) vanishes, and hence that the real part of the right-hand side of (5.5) is zero. Similarly, we can define $\mathbf{A} = (u, -v)$ and apply (5.4) and the other Cauchy-Riemann relation to conclude also that the imaginary part of (5.5) is zero — hence proving Cauchy's Theorem.

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1. For complicated paths such as shown in Figure 5.2, where it is not always obvious which is the anticlockwise direction, the positive direction is defined as the direction such that the interior of the circuit lies on the left of the path.

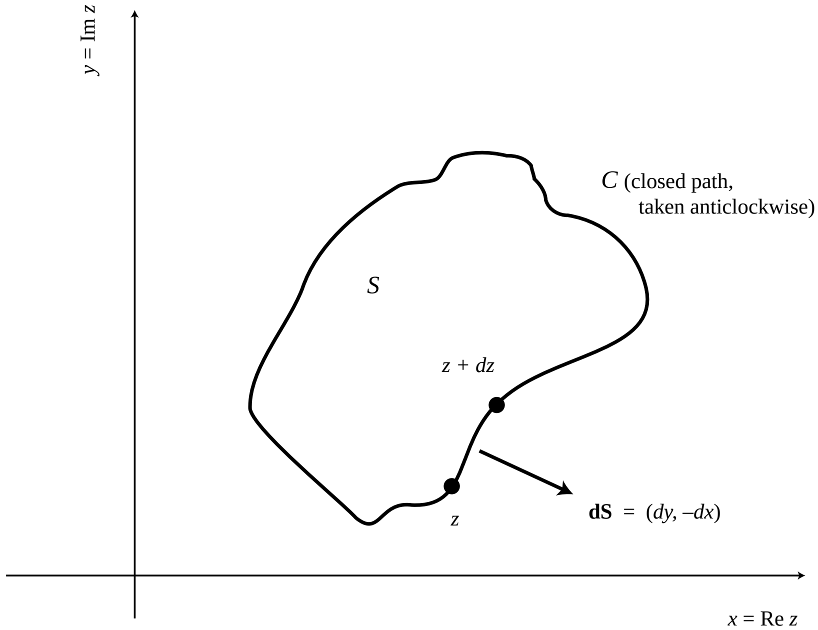

FIGURE **5.1**
A closed curve C is shown in the complex z-plane. For points z and $z + dz$ lying on C, dz is a line element and the outward normal to C, having magnitude $|dz|$ and direction perpendicular to (dx, dy), is then $(dy, -dx)$.

5.1.1 SOME OBVIOUS CONSEQUENCES OF CAUCHY'S THEOREM

Here are a couple of corollaries:

(i) If we wish to evaluate the integral of f with respect to z along a path L_1 from z_A to z_B, then we can choose an alternate path from z_A to z_B, say L_2, and get the same result — provided f is analytic on L_1 and L_2 and f is also analytic throughout the region between these two paths. That is,

$$\int_{z_A}^{z_B \text{ via } L_1} f\,dz = \int_{z_A}^{z_B \text{ via } L_2} f\,dz. \tag{5.7}$$

This result follows from (5.2), defining C to be the closed path from z_A to z_B via L_1, and then back to z_A via L_2 in reverse. It follows that we are free to choose any path, to evaluate the integral from z_1 to z_2. In particular, we can choose a path that helps to carry out the evaluation.

(ii) If we wish to evaluate the integral of f with respect to z on a closed path C, then we can choose an alternate closed path C' (see Figure 5.2) that is entirely inside C, provided f is analytic on C' and throughout the area enclosed between C and C'. This result is useful if f is known not to be analytic within a subregion of the interior of C that can be localized to within C'. The result is proved by considering the closed path defined as follows: starting at the point A (see Figure 5.2), go anticlockwise

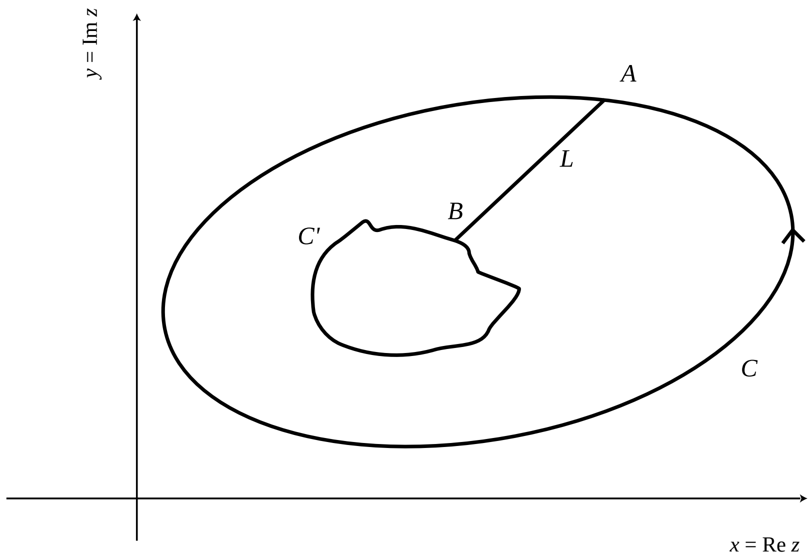

FIGURE **5.2**
A closed curve C is shown in the complex z-plane, and inside C is another closed path, C'.

around C, then from A to B on C' by a line L, than around C' *clockwise*, and finally from B back to A by the line in reverse. This whole path constitutes a closed circuit for which the interior is always on the left, and f is analytic in this interior. Note that we are not requiring f to be analytic everywhere inside C, only between C and C' and on the path itself. So, from (5.2),

$$\oint_{C+L-C'-L} f\ dz = 0.$$

The signs here, for L and C', indicate the direction taken. The two L contributions cancel, so $\oint_{C-C'} = 0$, which we can rewrite as

$$\oint_C f\ dz = \oint_{C'} f\ dz. \tag{5.8}$$

It is characteristic of the theory of complex functions of a complex variable, that it has results like (5.7) and (5.8) which are useful and simple to state, but which often take a long time to explain the first time you see them.

5.2 Singularities of $f(z)$, and the "Calculus of Residues"

If $f(z)$ behaves like $\dfrac{\lambda}{(z-z_0)^N}$ (where λ is a non-zero constant), for values of z near z_0, then the derivative of f does not exist at z_0, and f cannot be analytic there. f is said to have an Nth order singularity (sometimes called an Nth order pole) at $z = z_0$.

If z_0 is *outside* C it is still true that $\oint_C f(z)\,dz = 0$ (provided f is analytic on C and inside C). But what can we say about $\oint_C f\,dz$, if z_0 is *inside* the path of integration?

First, we note that $f(z)$ can be expanded in a series of powers of $(z - z_0)$, called a *Laurent series*, as follows:

$$f(z) = \sum_{n=-N}^{\infty} a_n (z - z_0)^n \tag{5.9}$$

where $a_{-N} = \lambda$. This is similar in some ways to a Taylor series, which would start with $n = 0$. But instead of the usual Taylor series, we must start the series at $n = -N$ to include the singularity that will dominate the behavior of $f(z)$ for values of z near z_0. (As we shall see, the Laurent series is useful because we can integrate it term by term. All the terms except one will vanish after the integration.)

Second, we can evaluate $\oint_C f(z)\,dz$, where z_0 lies inside C, by replacing C by a small circle of radius R and centered on z_0 itself. (This is because we can call this circle C' and apply (5.8).) Thus,

$$\oint_C f(z)\,dz = \oint_{C' = \text{ circle of radius } R, \text{ centered on } z_0} f(z)\,dz \tag{5.10}$$

Third, we note that

$$\oint_{C'} a_n (z - z_0)^n\,dx = a_n \int_0^{2\pi} R^n e^{in\theta} i R e^{i\theta}\,d\theta = i a_n R^{n+1} \int_0^{2\pi} e^{i(n+1)\theta}\,d\theta \tag{5.11}$$

(taking $z = Re^{i\theta}$, so that $dz = iRe^{i\theta}\,d\theta$). But

$$\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta = \frac{1}{i(n+1)} e^{i(n+1)\theta}\Big|_{\theta=0}^{\theta=2\pi} = 0, \text{ unless } n + 1 = 0 \text{ (i.e. } n = -1) \tag{5.12}$$

This last result follows because in general $n + 1$ is an integer different from zero, and $e^{i(n+1)2\pi} = \mathrm{e}^0 = 1$ ($e^{i\theta}$ is periodic with period 2π). In the special case $n = -1$, when we cannot divide by $n + 1$, we have from (5.11) that

$$\oint_{C'} \frac{a_{-1}}{z - z_0} dz = i a_{-1} \int_0^{2\pi} d\theta = 2\pi i a_{-1}. \tag{5.13}$$

Putting these results together, (5.9) through (5.13), we find that

$$\oint_C f(z)\,dz = 2\pi i a_{-1}. \tag{5.14}$$

The value a_{-1} is called the *residue* of f. This is a natural word to use, in the sense that once f has been integrated, the only term in the Laurent series that is left, is the one with coefficient a_{-1}. (A residue is something that is left after everything else has been removed.) Note that (5.14) is valid, even in situations where $f(z)$ has a singularity at $z = z_0$ that is even stronger than $\dfrac{1}{z - z_0}$. In other words, (5.14) applies even when $-N < -1$ in (5.9).

The only term that matters, is the so-called simple pole. The only residue, is that coming from the coefficient of $\dfrac{1}{z-z_0}$.

We can generalize (5.14) to the case that there is more than one place inside C where f is singular, by adding the contribution from each residue to obtain the total:

$$\oint_C f(z)\,dz = 2\pi i \times \text{ the sum of residues inside } C. \tag{5.15}$$

Next, suppose that $f(z)$ is analytic both on and inside C, so that $\oint_C f(z)\,dz = 0$. What then is the value of $\oint_C \dfrac{f(z)}{z-z_0}\,dz$? We can answer the question by applying to the function $g = \dfrac{f}{z-z_0}$ the results obtained above for a function with a pole.

The integrand g in this case is *not* analytic because it has a first order pole at $z = z_0$ (alternatively called a simple pole).

If z_0 is outside C, then the integral is zero because the singularity lies outside and the conditions for Cauchy's theorem are satisfied. But what is $\oint_C \dfrac{f(z)}{z-z_0}\,dz$ if z_0 lies inside C?

In this case we can write the Laurent series for the integrand as

$$g(z) = \frac{f(z)}{z-z_0} = \frac{f(z_0)}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n. \tag{5.16}$$

Replacing $f(z)$ in (5.14) by $g(z)$ and using (5.16) to identify the residue, we see that

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\ f(z_0). \tag{5.17}$$

This result is useful, because it tells us that we can evaluate f at any point inside C, once we know the values of $f(z)$ on the curve C itself. [All we have to do is turn (5.17) around and write it as $f(z_0) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{z-z_0}\,dz$, and then carry out the integration to obtain $f(z_0)$ for any point z_0 inside C.] Thus, the information about f on C itself is enough to determine values of f everywhere inside C! The result is remarkable, because it allows us to extrapolate from information about f provided on a *line*, to obtain information about f throughout an *area*. (Remember, in this application, f has to be analytic inside and on C, and z_0 has to be inside C. If z_0 is outside C, the integration in (5.17) gives zero.)

The calculus of residues (the name given to applications of (5.10), (5.14) – (5.17)) can be used to evaluate integrals. As a simple example, we can examine

$$I(a) = \int_{-\infty}^{\infty} \frac{dk}{k^2+a^2}. \tag{5.18}$$

The method of evaluation of integrals such as (5.18) is via a series of standard tricks, each of which takes some time to explain the first time you see it. But with a little practice, these steps can be applied very quickly so that the evaluation of the integration can often be written straight down, or obtained with minimal sidework.

Note in (5.18) that a is real and k ranges over all real values from $-\infty$ to $+\infty$, so that

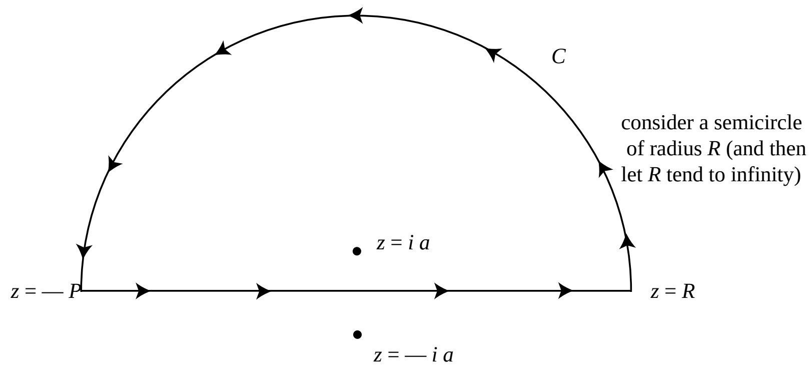

FIGURE **5.3**
This shows a particular path of integration, C, in the complex z plane; namely, a semicircle with diameter along the real axis (x values from $x = -R$ to $x = +R$), and an arc in the upper half plane (where z has a positive imaginary part). One of the poles of $\dfrac{1}{z^2 + a^2}$ lies inside C, the other lies outside. Note that C is taken in the positive direction (anticlockwise).

$I(a)$ must be real. Now we shall evaluate $I(a)$ by a method that extends k values to the complex plane.

The first step is to recognize that we can write (5.18) as

$$I(a) = \lim_{R \to \infty} \oint_C \frac{dz}{z^2 + a^2} \tag{5.19}$$

where C is the semicircular path shown in Figure 5.3. Note that C is composed of two parts: the original path in (5.18), running over all real values from $-\infty$ to $+\infty$; and a semicircle in the upper half plane. This second part turns out to contribute nothing to the circuit integral, because the integrand is so small on the large arc. The whole point of adding this semicircular path is that it completes the circuit. [2] But now, with (5.19), we can take the second step and use (5.15) to write

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2. More formally in proving (5.19), we see that in the limit as $R \to \infty$, the diameter of the semicircle becomes the real axis from $-\infty$ to $+\infty$, which is the extent of the integration in (5.18). Additionally we have a contribution from the large semicircular arc in the upper half of the complex z plane, described by $z = Re^{i\theta}$, for θ running from 0 to π; and then we have to take the limit of the contribution to the integral that comes from the semicircle, as $R \to \infty$. But this addition to the original path contributes nothing to the integration, because the integrand is so small on the large arc. In effect we have added zero to (5.18) to obtain (5.19). But now we have a circuit integral that can be evaluated by use of residues.

$$I(a) = 2\pi i \times \text{ the sum of residues of } \frac{1}{z^2 + a^2} \text{ inside } C$$

where now in the limit as $R \to \infty$ we must include any residue in the whole region above the real z-axis.

The third step is to evaluate the residues inside the circuit. In the present case this is simple, because there is only one such residue. To see this, note that the integrand here can be written as $\dfrac{1}{(z+ia)(z-ia)}$ so there is a pole inside the closed curve, at $z = +ia$. The other pole, at $z = -ia$, lies below the real z-axis and is therefore outside C.

The residue of $\dfrac{1}{z^2+a^2}$ at $z = +ia$ is, by definition, the coefficient of $\dfrac{1}{z-ia}$ in the Laurent expansion for $\dfrac{1}{z^2+a^2}$ in the vicinity of $z = +ia$. To obtain this residue, we note that in the vicinity of $z = ia$,

$$\frac{1}{z^2+a^2} \sim \frac{1}{z-ia}\left(\frac{1}{z+ia}\bigg|_{z=ia}\right) = \frac{1}{z-ia} \times \frac{1}{2ia}.$$

The residue is therefore $\dfrac{1}{2ia}$. Hence by evaluating the residue we obtain our answer:

$$I(a) = \int_{-\infty}^{\infty} \frac{dk}{k^2+a^2} = 2\pi i \times \frac{1}{2ia} = \frac{\pi}{a}. \tag{5.20}$$

As an alternative but related approach to evaluating (5.19) we can apply (5.16) and (5.17) with $g(z) = \dfrac{1}{z^2+a^2}$, $z_0 = ia$, and $f(z) = \dfrac{1}{z+ia}$. Then $f(z_0) = 1/(2ia)$, and again we obtain (5.20).

With a little practice, results such as (5.20) can be written down almost immediately without having to do any intermediate algebraic manipulations. Note that the first step in the evaluation of (5.18) requires some creativity, in order to find a path with the two properties that (a) it "completes the circuit" (allowing us to evaluate the integral by use of residues), and (b) that it adds nothing to the original integration. In the present case, a semicircle in the lower half plane would work just as well.

The same general approach allows us to see how to evaluate the inverse transform of $F(k) = \dfrac{1}{k^2+a^2}$. Inverting from k to x, F transforms back to $f(x)$ where x is real and

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}\,dk}{k^2+a^2} = \frac{1}{2\pi} 2\pi i \times \frac{e^{-ax}}{2ia} = \frac{e^{-ax}}{2a} \quad \text{(assuming } x > 0\text{)}.$$

In the evaluation of this integral by the calculus of residues, use of a semicircle in the upper half plane is appropriate because e^{ikx} is exponentially small when k has a positive imaginary part. (The semicircle in the lower half plane would be inappropriate because then e^{ikx} would be exponentially large.) A similar example of the calculus of residues is used elsewhere in these class notes to evaluate one of the inverse integral transforms needed to obtain the temperature associated with dyke injection (see Chapter 6).