CHAPTER 7

Solid angle, 3D integrals, Gauss's Theorem, and a Delta Function

We define the solid angle, subtended at a point P by a surface area S, to be

$$\Omega = \int_S \frac{\mathbf{r} \cdot \mathbf{dS}(\mathbf{r})}{r^3}. \tag{7.1}$$

In this integral, $\mathbf{dS}$ is chosen on the side of S that typically makes $\mathbf{r} \cdot \mathbf{dS}$ positive, as shown in Figure 7.1.

With this definition we can show that Ω depends only on the position of P with respect to the *perimeter* of S, because different surfaces S and S' that have the same perimeter will subtend the same solid angle (provided P does not lie inside the closed surface formed by S and S').

We are claiming here that

$$\Omega = \int_S \frac{\mathbf{r} \cdot \mathbf{dS}(\mathbf{r})}{r^3} = \int_{S'} \frac{\mathbf{r} \cdot \mathbf{dS}'(\mathbf{r})}{r^3}. \tag{7.2}$$

To prove this last result, consider the closed surface Σ formed by S and S'. Typically, one of the vectors $\mathbf{dS}$ and $\mathbf{dS}'$ will point out of Σ, and the other will point into Σ. As drawn in Figure 7.2, $\mathbf{dS}$ points in and $\mathbf{dS}'$ points out. If the direction of $\mathbf{d\Sigma}$ is defined everywhere to point out of the closed surface Σ, it follows that

$$\int_\Sigma \frac{\mathbf{r} \cdot \mathbf{d\Sigma}}{r^3} = \int_{S'} \frac{\mathbf{r} \cdot \mathbf{dS}'}{r^3} - \int_S \frac{\mathbf{r} \cdot \mathbf{dS}}{r^3}.$$

But, by Gauss's theorem,

$$\int_\Sigma \frac{\mathbf{r} \cdot \mathbf{d\Sigma}}{r^3} = \int_V \nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) dV,$$

where V is the volume inside the closed surface Σ, and the point P lies outside V.

It is easy to show (see Box 7.1) that $\nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right)$ is zero everywhere inside V, hence $\int_\Sigma \frac{\mathbf{r} \cdot \mathbf{d\Sigma}}{r^3} = 0$, and we can indeed use either S or S' in the definition of Ω.

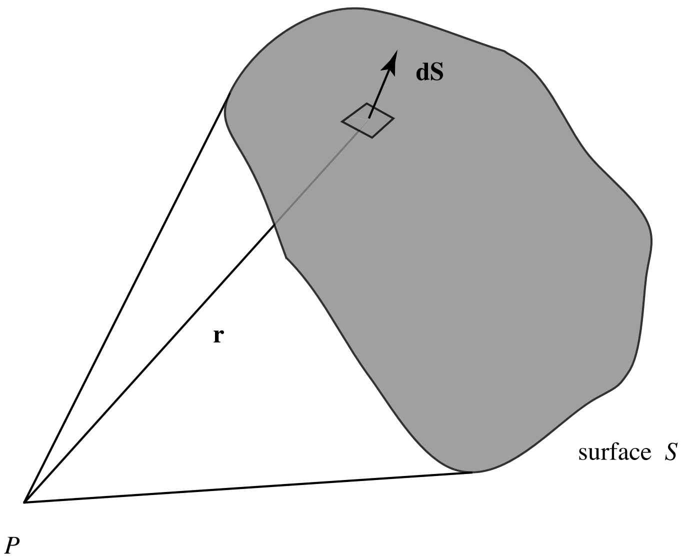


FIGURE **7.1**
This crudely illustrates a 3D relationship between the point P and the surface S. In general neither the surface nor its perimeter lies in a plane.

Ω is also equal to the area on the unit sphere (i.e., a sphere with unit radius, centered on P) cut by the straight lines from P to the perimeter of S. To see this, let S' be the surface shown in Figure 7.3 and let A be the area cut on the unit sphere.

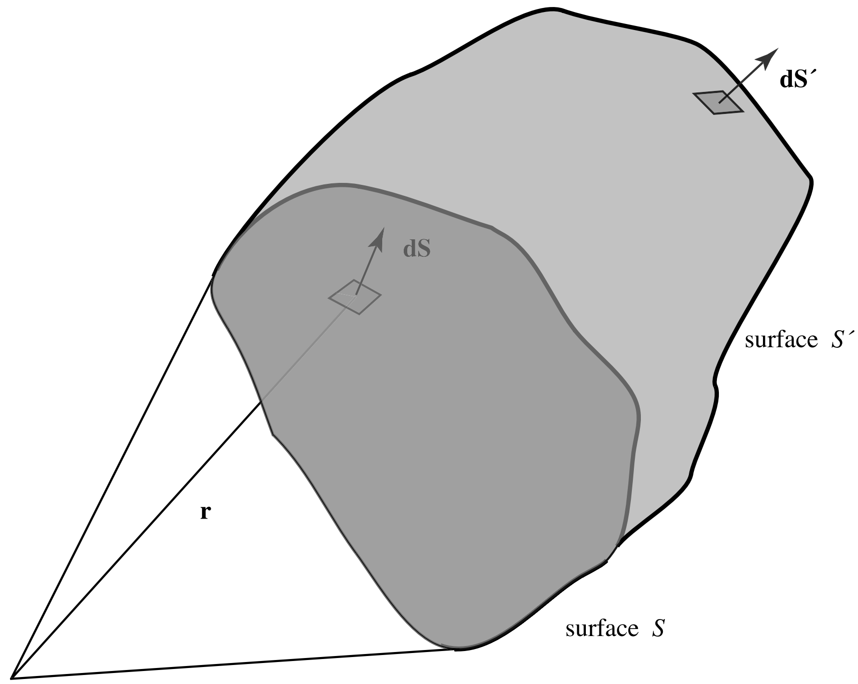
Then since $\Omega = \int_{S'} \frac{\mathbf{r}\cdot\mathbf{dS}'}{r^3}$ from (7.2), and S' is made up of area A plus a surface for which $\mathbf{r}$ and $\mathbf{dS}'$ are perpendicular, it follows that $\Omega = \int_A \frac{\mathbf{r}\cdot\mathbf{dA}}{r^3}$. But we can simplify further, since $\mathbf{r}$ is a unit vector on A, and is normal to $\mathbf{dA}$. We find

$$\Omega = \int_A dA = A. \tag{7.3}$$

It is interesting next to see what happens if instead of choosing two surfaces S and S' as in Figure 7.2, we choose S and S'' as in Figure 7.4. The two surfaces still form a closed surface, but now with P inside. We still take Σ (the sum of two surfaces, here S and S'') as the closed surface, but now $\mathbf{d\Sigma}$, $\mathbf{dS}$, and $\mathbf{dS}''$ are all outward-pointing vectors.

Recall from (7.3) that $\Omega = \int_S \frac{\mathbf{r}\cdot\mathbf{dS}}{r^3} = A$, where A is the area cut on the unit sphere. The total spherical area is 4π, so $\int_{S''} \frac{\mathbf{r}\cdot\mathbf{dS}''}{r^3} = 4\pi - A$, and $\int_S + \int_{S''} = \int_\Sigma = 4\pi$. But it is still true that $\int_\Sigma = \int_V \nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right) dV$, so why is this last integral not zero, as in the previous analysis when P was outside V? And why is it not zero, given the results developed in Box 7.1?

The reason is that the proof of $\nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right) = 0$, given in Box 7.1, fails at the place where $r = 0$, i.e. at the point P itself.



P

FIGURE **7.2**
Two surfaces, **S** and **S**′, with the same perimeter.

We have now proved two very important properties of the expression $\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right)$. It is zero everywhere except at $r = 0$ (as shown in Box 7.1). And the volume integral $\int_V \nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) dV$ is either zero or 4π, depending on whether the point P lies outside or inside the volume. Therefore the integrand must be a delta function. In fact, it must be 4π times the standard Dirac delta function in three-dimensional space.

The mathematical units in which a solid angle Ω is measured are called "steradians" (compare with radians, the mathematical units for ordinary angle). The solid angle representing the totality of all directions away from a point is 4π steradians (compare with the value 2π radians for the ordinary angle corresponding to all directions away from a point in a plane). The steradian is physically a dimensionless unit (just as radians and degrees are dimensionless).

Finally, note that $\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) = -\nabla^2 \left(\frac{1}{r}\right)$. We have therefore shown by our discussion of solid angles that

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\,\delta(\mathbf{r}),$$

BOX **7.1**

Two properties of a particular vector related to gravitation

At the heart of our discussion of solid angle is the vector

$$\frac{\mathbf{r}}{r^3}.$$

First we shall evaluate the divergence of this vector, and then we'll relate the vector to the gradient of a scalar.

Provided $r \neq 0$, we can write

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) = \frac{\partial}{\partial x_i}\left(\frac{x_i}{r^3}\right)$$
$$= \frac{3}{r^3} - 3\frac{x_i}{r^4}\frac{\partial r}{\partial x_i}. \tag{1}$$

But $r^2 = x_j x_j$, and differentiating this result with respect to x_i we obtain $2r\frac{\partial r}{\partial x_i} = 2x_j\frac{\partial x_j}{\partial x_i} = 2x_j\delta_{ij} = 2x_i$, and hence

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}. \tag{2}$$

It follows from (1) and (2) that

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) = \frac{3}{r^3} - 3\frac{x_i}{r^4}\frac{x_i}{r} = \frac{3}{r^3} - 3\frac{r^2}{r^5} = 0. \tag{3}$$

Next, we can note that the vector $\frac{\mathbf{r}}{r^3}$ is related to the gradient of the scalar $\frac{1}{r}$. This follows because, using components in a cartesian system,

$$\nabla\left(\frac{1}{r}\right)\Bigg|_i = \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x_i} = -\frac{x_i}{r^3} = -\frac{\mathbf{r}}{r^3}\Bigg|_i,$$

and so

$$-\frac{\mathbf{r}}{r^3} = \nabla\left(\frac{1}{r}\right). \tag{4}$$

Note that $\frac{\mathbf{r}}{r^3} = \frac{\text{unit vector in the direction of } r \text{ increasing}}{r^2}$. Therefore, in the context of gravity theory, we can recognize the vector $-\frac{\mathbf{r}}{r^3}$ as having magnitude and direction like those of an inverse square law for an attractive force between two particles a distance $\mathbf{r}$ apart. Equation (4) gives the associated potential.

Putting the results (3) and (4) together, we find that

$$\nabla^2\left(\frac{1}{r}\right) = 0 \qquad \text{provided} \qquad r \neq 0. \tag{5}$$

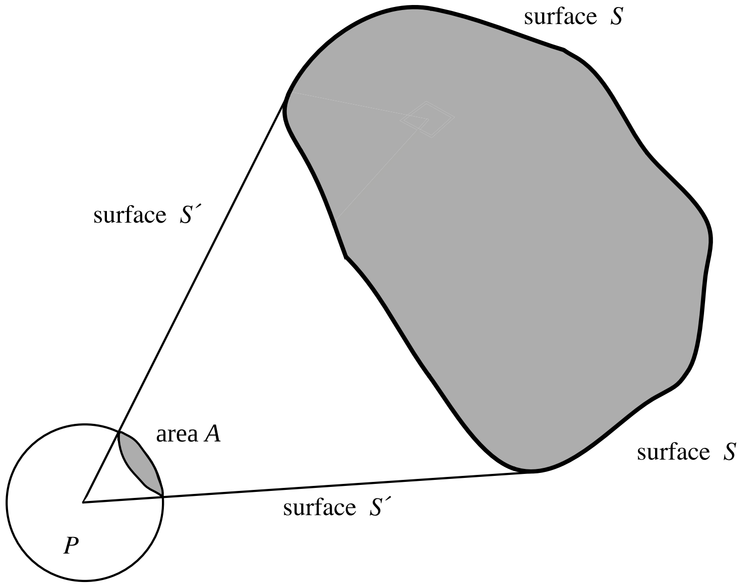


FIGURE **7.3**
Solid angle is shown as an area A projected from S onto part of the unit sphere. The area S' is made up from area A plus the part of a cone between the perimeter of A and the perimeter of S.

where $\delta(\mathbf{r})$ is the three-dimensional Dirac delta function.

More generally,

$$\nabla^2\left(\frac{1}{|\mathbf{x}-\boldsymbol{\xi}|}\right) = -4\pi\,\delta(\mathbf{x}-\boldsymbol{\xi}).$$

Problems

7.1 What is the solid angle subtended by the blackboard in Schermerhorn Room 555 in the following cases: (a) at the eye of a person in the middle of the main row where people sit; (b) at the eye of a person in a far corner of the room; and (c) at the eye of a myopic (short-sighted) instructor with his or her nose right up against the board? (Give approximate answers, in steradians.)

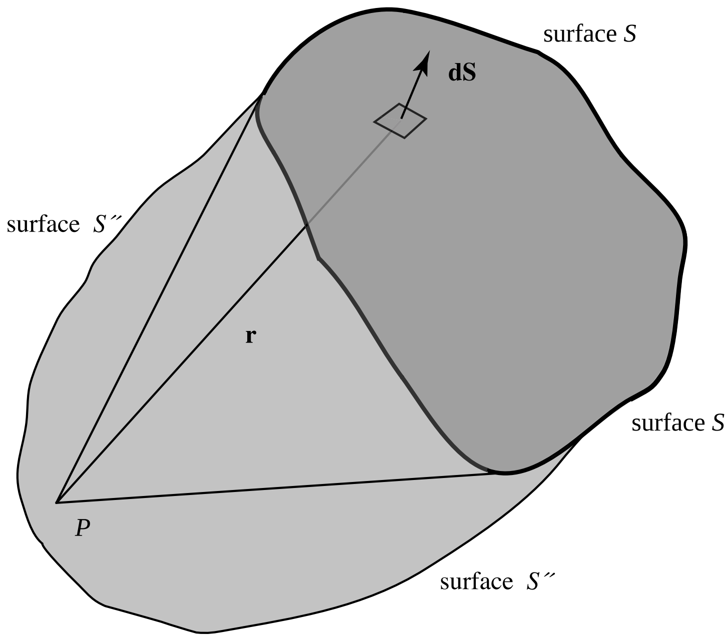


FIGURE **7.4**
P is now inside the closed surface formed by **S** and **S**″ (which share a common perimeter).