CHAPTER 8

Lagrange Multipliers, and Two Applications in the Study of Shear Fracture

This chapter describes the use of Lagrange multipliers to find maxima and minima of a function of a set of variables, in the case that the variables cannot freely take on any value, but are subject to one or more additional constraints.

The basic method is given, followed by two applications in the study of shear stress at a point in a generally stressed solid. Thus, we shall answer the questions:

(i) What is the plane of maximum shear stress? (See Section 8.2.)

(ii) What is the plane of maximum Coulomb stress? (See Section 8.3.)

8.1 The Basic Method

First let's consider

Problem A: how do we solve for the values of $(x_1, x_2, \ldots)$ that give maximum values of a function $G = G(x_1, x_2, \ldots)$, subject to the constraint that $H(x_1, x_2, \ldots) = 0$?

The constraint is sometimes called a *side condition*. An example in three dimensions would be the requirement that $\mathbf{x}$ is a unit vector. Then the side condition would be $H(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2) - 1$.

Note an example of Problem A for two dimensions shown in Figure 8.1 in which there is an obvious graphical solution. We simply find the extreme values of G, as $\mathbf{x}$ moves over the ellipse $ax_1^2 + 2bx_1x_2 + cx_2^2 = 1$. Later, we can check if the extrema are maxima or minima.

It is apparent from Figure 8.1 that the solutions we want are the points where the contours of G are tangential to the curve $H = 0$. These are points marked as P and Q in the Figure. Other points on $H = 0$, between P and Q, lie between these extrema. If G is a more complicated function of (x_1, x_2) then there may be more than two extreme values of G as x_1 and x_2 vary on the ellipse. But, any maximum of G (subject to the side condition $H = 0$) must be one of them.

This graphical presentation of properties of the solution enables us to set up a different calculus problem, which we can call Problem B, which is easy to solve directly, and which has the same solutions as Problem A. In this way we can solve Problem A indirectly.

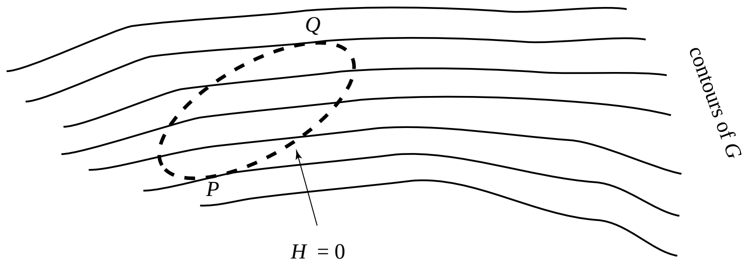

FIGURE **8.1**
This shows a two-dimensional example of the main idea behind the method of Lagrange multipliers. Each point of the diagram represents a value of the vector $\mathbf{x} = (x_1, x_2)$. The set of points for which $H(\mathbf{x}) = 0$ in this case is an ellipse (and if we were using an independent variable such as $\mathbf{x} = (x_1, x_2, x_3)$ or $\mathbf{n} = (n_1, n_2, n_3)$ with three scalar variables, then $H = 0$ would be a surface such as a sphere). Seven contour lines for G are shown. On each of them, G is equal to a constant. The constant varies from one contour to another.

Thus, at points P and Q in Figure 8.1, note that the normals to the contours of G are parallel to the normals of the ellipse (i.e. the direction of the vector whose components are $\frac{\partial G}{\partial x_i}$ is parallel to the direction whose components are $\frac{\partial H}{\partial x_i}$ $(i = 1, 2)$. So, the points P and Q have $\mathbf{x}$ values that satisfy

$$\frac{\partial G}{\partial x_i} + \lambda \frac{\partial H}{\partial x_i} = 0 \quad (i = 1, 2) \tag{8.1}$$

for some value of λ. This result suggests setting up, instead of Problem A, the following

Problem B: for fixed λ, how do we find values of $\mathbf{x}$ that maximize the function $F(\mathbf{x}) = G(\mathbf{x}) + \lambda H(\mathbf{x})$? In general the solutions will depend on λ. We can write them as $\mathbf{x} = \mathbf{x}(\lambda)$. Choose the value of λ for which $H(\mathbf{x}(\lambda)) = 0$.

To solve Problem B in the case of m variables x_i $(i = 1, 2, \dots, m)$, we set up $m + 1$ equations in $m + 1$ unknowns:

$$\frac{\partial G}{\partial x_i} + \lambda \frac{\partial H}{\partial x_i} = 0 \quad (i = 1, 2, \dots, m), \quad \text{and} \quad H = 0.$$

We claim that the solution to Problem B is also a solution to Problem A.

The (initially) unknown constant λ is called a *Lagrange multiplier*.

More generally, there may be several side conditions $H_j(\mathbf{x}) = 0 \quad (j = 1, 2, \dots, n)$ as well as a large number, m, of independent variables $(x_1, x_2, x_3, x_4, \dots)$. The extrema of a function $G(\mathbf{x})$, subject to these n side conditions, is found by working with n Lagrange multipliers. The solution is found from the m conditions for extrema of $G + \sum_{j=1}^{n} \lambda_j H_j$, namely (8.1) for $i = 1, 2, \dots, m$, and then choosing the n Lagrange multipliers λ_j so that

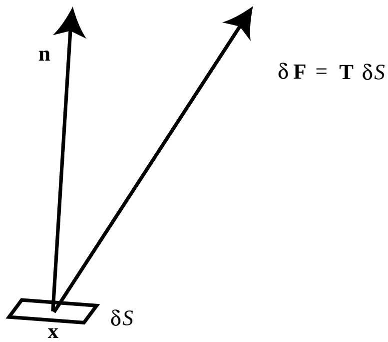

FIGURE **8.2**
This shows an area element centered on **x**, with scalar magnitude δS and direction given by the normal vector **n**; and the associated traction vector **T**. The traction is "force per unit area," and the total surface force acting across the area element is $\delta \mathbf{F} \propto \delta S$. The traction is the constant of proportionality between $\delta \mathbf{F}$ and δS. As shown in Figure 1.4, δS may be part of an internal surface. In this chapter we are using $\boldsymbol{\sigma}$ for the stress tensor, which is a common notation. In Chapter 1 we used $\boldsymbol{\tau}$. We showed in equation (1.18) and Section 1.1.1 that $T_i = \tau_{ij} n_j$, so in the present chapter the corresponding result is $T_i = \sigma_{ij} n_j$.

$H_j = 0$ for all j. This gives n more equations, for a total of $m + n$ equations for the $m + n$ unknowns.

8.2 Application to Finding Planes of Maximum Shear Stress

In a stress field with principal stresses $\sigma_1, \sigma_2, \sigma_3$, we use the principal axes of stress as coordinate axes. The usual rule $T_i = \sigma_{ij} n_j$, relating traction vector and normal vector, then reduces to $\mathbf{T} = (\sigma_1 n_1, \sigma_2 n_2, \sigma_3 n_3)$; and the normal stress $\sigma_n = \sigma_{ij} n_i n_j$ (see equation (1.19)) is given in this case by $\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$.

Resolving **T** into its normal (σ_n) and tangential (σ_t) components, the shear stress (σ_t) is therefore given as a function of components of **n** by

$$\sigma_t^2 = \mathbf{T} \cdot \mathbf{T} - \sigma_n^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - (\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)^2. \tag{8.2}$$

From the discussion of "Problem A" and "Problem B" in the previous section, we see now that the problem of finding maxima in σ_t^2 as **n** varies, subject to the constraint $n_1^2 + n_2^2 + n_3^3 = 1$, is equivalent to the problem of maximizing $F = \sigma_t^2 + \lambda(n_1^2 + n_2^2 + n_3^2)$ as each of (n_1, n_2, n_3) vary independently, and *then* imposing $n_1^2 + n_2^2 + n_3^3 = 1$.

To find the planes of maximum shear stress, we therefore have four equations for the three unknown scalar components of **n**, and the unknown Lagrange multiplier λ:

$$\frac{1}{2}\frac{\partial F}{\partial n_1} = \sigma_1^2 n_1(1 - 2n_1^2) - 2\sigma_1\sigma_2 n_1 n_2^2 - 2\sigma_1\sigma_3 n_1 n_3^2 + \lambda n_1 = 0,$$

plus two more such equations from $\dfrac{\partial F}{\partial n_2} = 0$, $\dfrac{\partial F}{\partial n_3} = 0$, plus the side condition

$$n_1^2 + n_2^2 + n_3^3 = 1 \tag{8.3}$$

.

From these four equations we can find

$$\sigma_1 n_1[+(n_2^2 + n_3^2 - n_1^2)\sigma_1 - 2n_2^2\sigma_2 - 2n_3^2\sigma_3] = -\lambda n_1,$$

$$\sigma_2 n_2[-2n_1^2\sigma_1 + (n_3^2 + n_1^2 - n_2^2)\sigma_2 - 2n_3^2\sigma_3] = -\lambda n_2,$$

and

$$\sigma_3 n_3[-2n_1^2\sigma_1 - 2n_2^2\sigma_2 + (n_1^2 + n_2^2 - n_3^2)\sigma_3] = -\lambda n_3.$$

There are three different solutions for **n**, and these are

$$\{n_1 = 0, \quad n_2^2 = n_3^2 = \frac{1}{2}, \quad \lambda = \sigma_2\sigma_3\} \tag{8.4}$$

or

$$\{n_2 = 0, \quad n_2^3 = n_1^2 = \frac{1}{2}, \quad \lambda = \sigma_3\sigma_1\} \tag{8.5}$$

or

$$\{n_3 = 0, \quad n_2^1 = n_2^2 = \frac{1}{2}, \quad \lambda = \sigma_1\sigma_2\}. \tag{8.6}$$

For each of these three solutions, we can go back to find the value of σ_t^2 at each extremum. Our formula (8.2) for σ_t^2 gives the following values:

$$\sigma_t^2 = [\frac{1}{2}(\sigma_2 - \sigma_3)]^2, \qquad \text{(for 8.4)}$$

$$\sigma_t^2 = [\frac{1}{2}(\sigma_3 - \sigma_1)]^2, \qquad \text{(for 8.5)}$$

$$\sigma_t^2 = [\frac{1}{2}(\sigma_1 - \sigma_2)]^2. \qquad \text{(for 8.6)}$$

If the principal stresses σ_1, σ_2, and σ_3 are all different, then one of the three extrema will be greater than the others. If we number axes so that

$$\sigma_3 < \sigma_2 < \sigma_1 < 0 \quad \text{(all compressions)}, \tag{8.7}$$

then the maximum shear stress has magnitude $\frac{1}{2}(\sigma_1 - \sigma_3)$ corresponding to solution (8.5). We might call this the“greatest maximum” or the “global maximum.” The other two maxima are lower in value. Sometimes, they are called “local maxima.” Each of the three

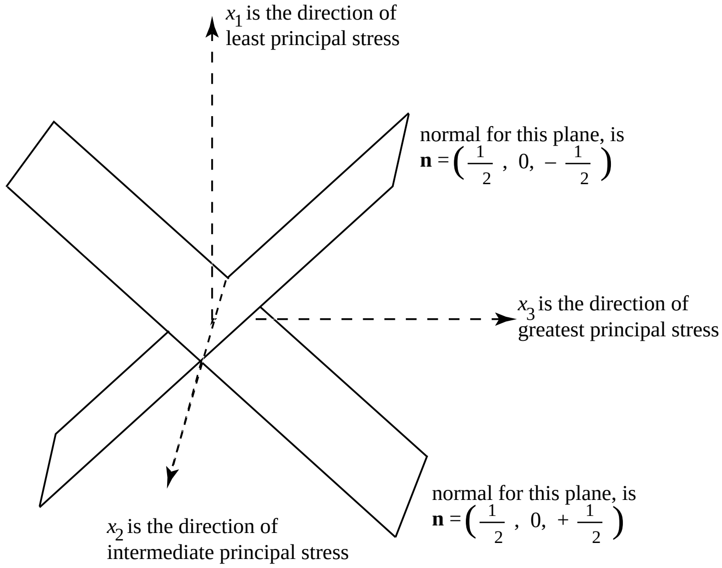

FIGURE **8.3**
Planes of greatest shear stress.

solutions (8.4) – (8.6) consists of a pair of planes, mutually at right angles. The particular solution which gives the greatest maximum is shown in Figure 8.3.

Note that the pair of planes of greatest shear stress contain the intermediate stress axis, and both planes lie at 45° to the directions of greatest and least principal stress.

8.3 The Effect of Friction

The plane that will eventually fail as stresses gradually increase, is not necessarily one of the two planes that has the greatest value of shear stress. Resistance to failure will come in part from a frictional force, which is proportional to normal stress (Coulomb's law). Thus, fault planes are more likely to develop on planes for which the difference between σ_t and $\mu\sigma_n$ is maximized. Here μ is a "coefficient of friction," and has value around unity for most material, including rock surfaces.

The only question remaining (in setting up a Lagrange multiplier problem) is whether we want extrema of $\sigma_t + \mu\sigma_n$, or $\sigma_t - \mu\sigma_n$ (as n varies). Especially, this is tricky if we are careful about thinking of traction and (compressive) pressure having opposite signs. To explain this, we again consider the case that all principal stresses are compressional, and the ordering of principal axes is as given by (8.7).

Figure 8.4 then shows a plane lying in the intermediate direction; so $n_2 = 0$. In this case, $\sigma_t^2 = \sigma_1^2 n_1^2 + \sigma_3^2 n_3^2 - (\sigma_1 n_1^2 + \sigma_3 n_3^2)^2$. Thus we can find a quite simple formula for the shear stress. It is given by

$$\sigma_t^2 = n_1^2 n_3^2 (\sigma_1 - \sigma_3)^2 \qquad \text{(using } n_1^2 + n_3^2 = 1\text{)}. \tag{8.8}$$

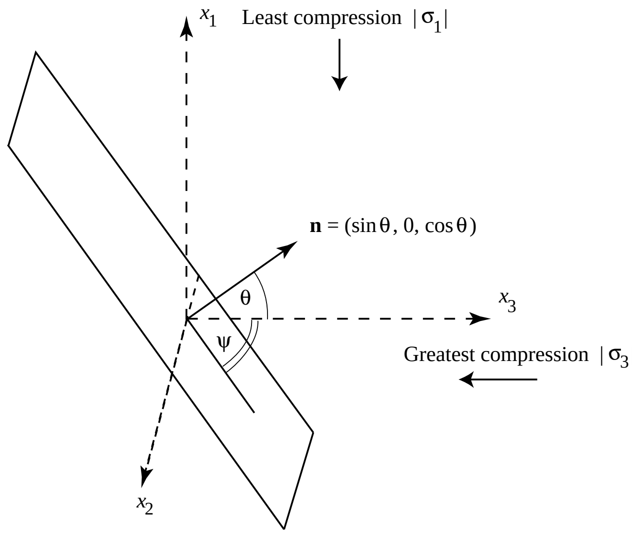

FIGURE **8.4**
A plane S containing the direction of the intermediate principal stress is shown. Its normal lies in the plane of the greatest and least compressive stresses, and equation (8.8) gives the shear stress on the plane S or different values of $\mathbf{n} = (n_1, 0, n_3) = (\cos\theta, 0, \sin\theta)$.

The forces per unit area on the block shown in Figure 8.5 are positive in the tangential and normal directions there shown, and so the quantity to be maximized is $|\sigma_t| - \mu|\sigma_n|$, which is $|\sigma_t| + \mu\sigma_n$. (Note that $\sigma_n = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$ and so σ_n is negative for compressive principal stresses.)

Hence, we wish to maximize the expression

$$G = \sqrt{\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - (\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)^2} + \mu(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2).$$

Introducing a Lagrange multiplier, we form $G + \lambda(n_1^2 + n_2^2 + n_3^2) = F$ and get 3 equations from $\dfrac{\partial F}{\partial n_i} = 0$ $(i = 1, 2, 3)$. These are:

$$\begin{aligned}
\sigma_1 n_1[+(n_2^2 + n_3^2 - n_1^2)\sigma_1 - 2n_2^2\sigma_2 - 2n_3^2\sigma_3] + 2\mu\sigma_1\sigma_t n_1 &= -2\lambda n_1\sigma_1 \\
\sigma_2 n_2[-2n_1^2\sigma_1 + (n_3^2 + n_1^2 - n_2^2)\sigma_2 - 2n_3^2\sigma_3] + 2\mu\sigma_2\sigma_t n_2 &= -2\lambda n_2\sigma_t \\
\sigma_3 n_3[-2n_1^2\sigma_1 - 2n_2^2\sigma_2 + (n_1^2 + n_2^2 - n_3^2)\sigma_3] + 2\mu\sigma_3\sigma_t n_3 &= -2\lambda n_3\sigma_t.
\end{aligned} \tag{8.9}$$

If $n_2 = 0$, then $\sigma_t = n_1 n_3(\sigma_1 - \sigma_3)$. From the first and third of the last set of three equations,

$$\begin{aligned}
(n_3^2 - n_1^2)\sigma_1 - 2n_3^2\sigma_3 + 2\mu n_1 n_3(\sigma_1 - \sigma_3) &= -\frac{2\lambda}{\sigma_1} n_1 n_3(\sigma_3 - \sigma_1) \\
-2n_1^2\sigma_1 + (n_1^2 - n_3^2)\sigma_3 + 2\mu n_1 n_3(\sigma_1 - \sigma_3) &= -\frac{2\lambda}{\sigma_3} n_1 n_3(\sigma_3 - \sigma_1).
\end{aligned} \tag{8.10}$$

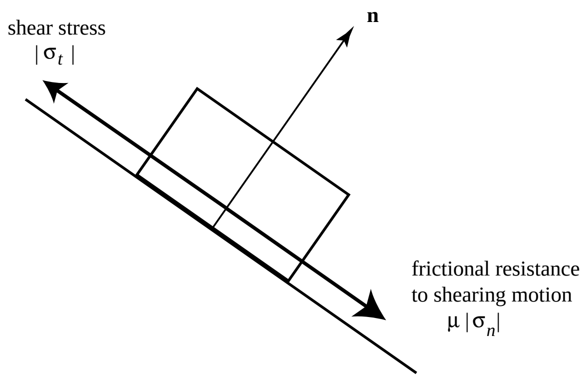

FIGURE **8.5**
To obtain an intuitive appreciation of the friction stress, a block is shown, held in place by frictional forces on an inclined plane. If the block is pushed to move in a way similar to motion on the fault shown in Figure 8.5, then it is subject to two tangential forces. One, is the shear stress $|\sigma_t|$. The other, is frictional resistance to shearing motion, which can reach up to $\mu|\sigma_n|$ in the opposite direction.

From (8.10) we can eliminate λ to find

$$(n_3^2 - n_1^2)(\sigma_3 - \sigma_1)^2 = -2\mu n_1 n_3(\sigma_3 - \sigma_1)^2$$

and hence we find that

$$\mu^{-1} = \frac{2n_1n_3}{n_1^2 - n_3^2}. \tag{8.11}$$

This last result can be turned into a straightforward trigonometrical equation for **n**, if we are given a value for μ. To see this, we note that $n_2 = 0$ and use the complementary angles θ and ψ of Figure 8.4, $\tan\theta = \dfrac{n_1}{n_3}$ and $\tan\psi = \dfrac{n_3}{n_1}$. It then follows from (8.11) that

$$\tan 2\psi = \frac{2\tan\psi}{1-\tan^2\psi} = \frac{1}{\mu}. \tag{8.12}$$

As an example: if $\mu = 1$, then $2\psi = 45^\circ$. So $\psi = 22\frac{1}{2}^\circ (\theta = 67\frac{1}{2}^\circ)$, and the plane is only $22\frac{1}{2}^\circ$ from the greatest compressive direction. The angle is much less than the 45° we found for the plane of maximum shear stress (see Section 8.2), because now we are taking friction into account and the frictional stress (which tends to prevent fault slip) is reduced by having a smaller angle between the fracture plane and the direction of greatest compressive stress.