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Enhanced sensitivity of persistent events to weak forcing in dynamical and stochastic systems: Implications for climate change

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Abstract. Low-dimensional models can give insight into the climate system, in particular its response to externally imposed forcing such as the anthropogenic emission of greenhouse gases. Here, we use the Lorenz system, a chaotic dynamical system characterized by two "regimes", to examine the effect of a weak imposed forcing. We show that the probability density functions (PDF's) of time-spent in the two regimes are exponential, and that the most dramatic response to forcing is a change in the frequency of occurrence of extremely persistent events, rather than the weaker change in the mean persistence time. This enhanced sensitivity of the "tails" of the PDF's to forcing is quantitatively explained by changes in the stability of the regimes. We demonstrate similar behavior in a stochastically forced double well system. Our results suggest that the most significant effect of anthropogenic forcing may be to change the frequency of occurrence of persistent climate events, such as droughts, rather than the mean.

1. Introduction

An important question in current research is the impact of anthropogenic forcing on the climate system. Most studies [*IPCC*,, 1996] have focused on changes in the mean of some variable, such as the mean temperature of the Earth. The mean, however, is not the only quantity of societal relevance, and we can also ask how might the fluctuations in the variables change. Motivated by recent studies drawing insight into possible climate change from simple nonlinear chaotic models [*Palmer*, 1999; *Corti et al.*, 1999], we too examine the response of simple models to simple forcing, but ask a different set of questions. We examine how the persistence of variables might change, and focus on an issue of potentially huge impact on society, namely the impact of forcing on the frequency of unusually persistent events (e.g., droughts and floods).

The real climate system is extremely complex, but can conceptually be thought of as a nonlinear dynamical system with preferred states or "regimes" [*Palmer*, 1999]. So in the spirit of *Palmer* [1999] we study the response to weak imposed constant forcing of a far simpler model, the Lorenz system. The Lorenz system [*Lorenz*, 1963] is governed by

$$\begin{aligned}
\dot{x} &= -\sigma x + \sigma y + f_o \cos\theta \\
\dot{y} &= -xz + rx - y + f_o \sin\theta \\
\dot{z} &= xy - bz
\end{aligned}$$

with $f_o = 0$. The forcing is that introduced by *Palmer* [1999], and we retain it here along with the conventional choice of coefficients ($r = 28$, $\sigma = 10$, $b = 8/3$). Figure 1 shows the projection of the state vector on the x-y plane for ($f_o = 2.5$, $\theta = 70°$). The system is characterized by chaotic oscillations around two unstable fixed points, thus defining two "regimes" (which we label C^+ and C^-), and irregular fluctuations between these regimes. With these parameters the system also has a third unstable fixed point at the origin. The unforced system is symmetric with respect to the two regimes (i.e., with $f_o = 0$, the equations are invariant under the transformation $x \to -x, y \to -y$).

2. Response to Weak Forcing

The response of the system to forcing can be studied in various ways. *Palmer* [1999] considered changes in the probability density function (PDF) of the time filtered (by a running average) state vector in the x-y plane as the forcing angle, θ, is varied. In the absence of forcing, the PDF's associated with the two regimes are equal. With an imposed forcing, there is an increase in the PDF associated with one regime, and a corresponding decrease in that of the other regime. Significantly, the positions in phase space of the PDF maxima (which nearly coincide with the location of the fixed points) do not change appreciably as either f_o or θ are varied. This is also seen in Figure 1, where the fixed points of the forced and unforced system are nearly identical. The implication for the real climate system, noted by *Palmer* [1999], is that anthropogenically forced changes in climate would project largely onto modes of natural climate variability, and thus the effect of anthropogenic forcing would manifest itself through changes in the frequency of occurrence of natural patterns of variability. The PDF of the state vector is sensitive to the filtering time scale [*Marshall and Molteni*, 1993]. So, instead, we examine the PDF of time spent (or persistence time, T_p) in a regime. To better discriminate between the two regimes we project the state vector onto the empirical orthogonal functions (EOF's) of the system (diagonal lines on Figure 1). The C^+ (C^-) regime is then defined by $x' > 0$ ($x' < 0$). Thus, T_p^+ (T_p^-) is the persistence time of the C^+ (C^-) regime, i.e., the duration between entering the regime and exiting it. Between the time the system enters a regime and leaves it, we also compute the minimum Cartesian distance (R_{min}) of the state vector to the fixed point corresponding to that regime.

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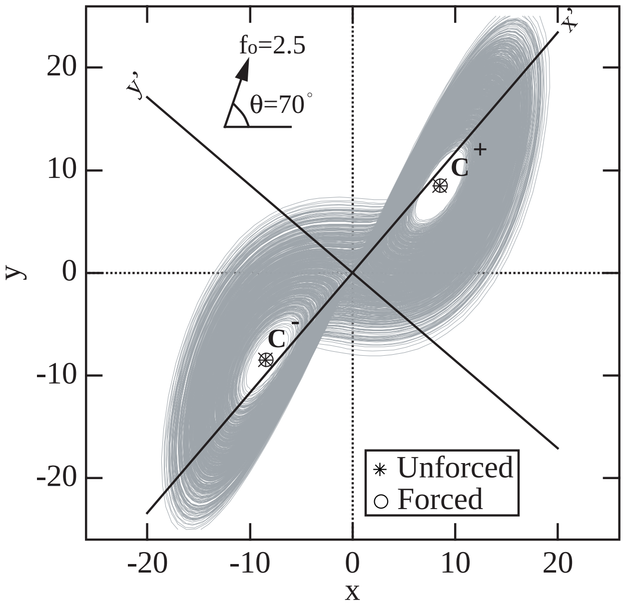

Figure 1. Projection of forced Lorenz system state vector on the x-y plane. Also shown is the forcing vector and the fixed points of the unforced ($*$) and forced ($\circ$) systems. Oscillations around the fixed points characterize the two regimes C^+ and C^-.

Figure 2a is a plot of T_p versus $\ln R_{min}$ for the unforced and forced ($f_o = 2.5$, $\theta = 60°$) systems. For segments of the trajectory which come sufficiently close to the fixed points there is a distinct difference between the persistence time of the two regimes. This can be understood in terms of the behavior of the system when linearized about the unstable fixed points. For the set of parameters chosen here the linearized system has three eigenvalues, one of which is real and negative, while the other two are complex ($\lambda \equiv \lambda_r \pm i\lambda_i$, $\lambda_r > 0$). Perturbations from the fixed point grow as $e^{\lambda_r t} e^{i\lambda_i t}$. This linear theory leads us to expect that the envelope of the distance from the fixed point increases as $R_{min} e^{\lambda_r t}$ until it reaches a critical value (R_{max}) when the system flips over to the other regime. Based on linear theory, then, the persistence time should scale as $T_p \sim \frac{1}{\lambda_r} \ln \frac{R_{max}}{R_{min}}$, which appears to be the case for $R_{min} < 8$. Furthermore, consistent with this idea, λ_r (C^+) $< \lambda_r$ (C^-), i.e., the fixed point associated with C^+ is less unstable than the C^- fixed point (for the particular forcing vector used). The imposed forcing breaks the symmetry between the two fixed points.

Figure 2b shows the PDF of persistence time for the two regimes. Evidently, the PDF of T_p is exponential ($\sim \exp(sT_p)$) and quite different for the two regimes. Note that most of the "events" have short persistence times, and it is these short lived events which dominate the mean of T_p ($< T_p >$). Furthermore, the relative difference between the two regimes for these short lived (but frequently occurring) events is quite small. In contrast, there are large differences between the two regimes in the frequency of occurrence of the more persistent events. These changes in the PDF of persistent events are not related to the PDF of R_{min}. Since, as seen in Figure 2c, the PDF's of R_{min} for the two regimes are nearly identical for R_{min} <11, this shows that the enhancement in the frequency of persistent events is not due to an increase in the probability of a closer approach to the fixed points. Rather, as we will explore further below, it is due to a change in stability of the regimes.

To quantify the sensitivity to imposed forcing of the frequency of occurrence of persistent events, we have computed the slope (s) of the PDF of T_p (on a log-linear plot as in Figure 2b) for a range of forcing angles and amplitudes. The upper panel in Figure 3 shows that the slope s is directly related to λ_r, the linear stability at the fixed points. The slope gets shallower (higher probability of persistent events) as λ_r becomes smaller (fixed point is less unstable). The large changes in slope as the forcing angle and amplitude are varied demonstrate that we can get orders of magnitude enhancement of the probability of an extremely persistent event, even though the mean changes only by $O(10\%)$ (lower panel of Figure 3).

The addition of forcing perturbs the location of the fixed points and since the Jacobian of the linearized system is a function of the point in phase space about which the linearization is performed, the eigenvalues of the Jacobian change. In particular, slight perturbations in the position of the fixed points can lead to $O(1)$ changes in the real part of the eigenvalues associated with those fixed points (see Figure 3), and thus to the observed extreme sensitivity of the probability of persistent events. In contrast, changes in the imaginary part of the eigenvalues are relatively very small. Our results therefore suggest that the systems' most significant response, the change in the PDF of persistence time,

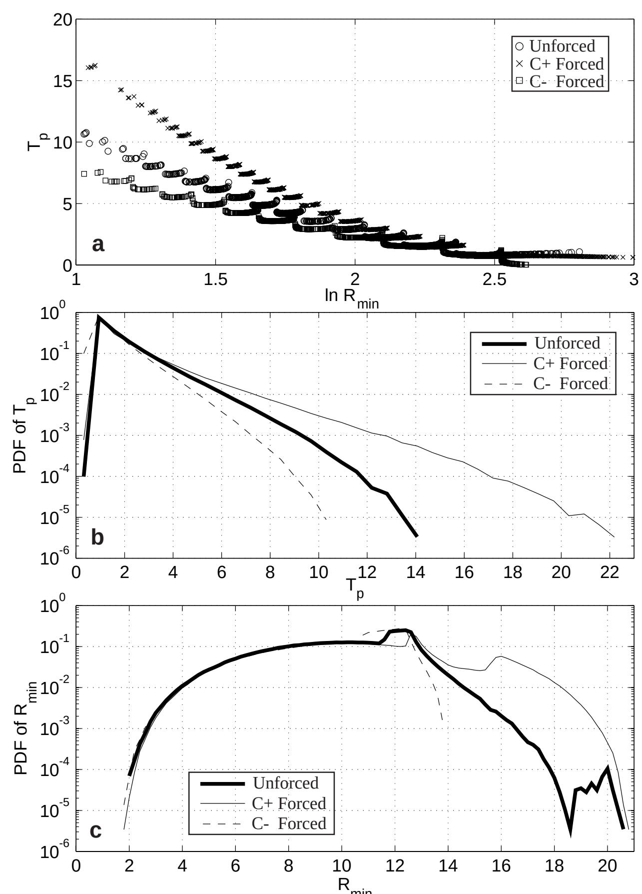

Figure 2. Plots of (a) T_p versus $\ln R_{min}$, (b) PDF of T_p, and (c) PDF of R_{min} for the unforced and forced ($f_o = 2.5$, $\theta = 60°$) Lorenz system.

is a direct consequence of changes in the linear stability of the fixed points, and is dramatically more sensitive than the relatively small change in the mean.

3. Comparison With a Stochastic Process

Finally, we compare the behavior of the Lorenz system with that of a simple stochastic process, the motion of a particle in a double well potential, $U(x)$, subject to a fluctuating force. The system is governed by the stochastic differential equations [*Gardiner*, 1985]

$$\begin{aligned} dx &= vdt \\ dv &= -U'(x)dt - \beta vdt + D^{1/2}N(t)(dt)^{1/2} \end{aligned}$$

where $N(t)$ is a zero-mean, unit-variance, normally distributed random variable that is statistically independent of $N(t')$ for all $t' \neq t$. The potential $U(x) = x^4 - 2x^2 - f_o x$ is symmetric about $x = 0$ when $f_o = 0$ (inset, Figure 4a). The deterministic system ($D \equiv 0$) with $f_0 = 0$ has stable equilibria at $x = \pm 1$ and an unstable equilibrium at $x = 0$. The addition of stochastic forcing ($D > 0$) allows the system to jump from one stable equilibrium to the other. The stationary solution of the Fokker–Planck equation shows the marginal PDF of x to be bimodal ($\sim \exp(-2\beta U(x)/D)$. To compare this process with the Lorenz system, we again define two states, C$^+$ ($x > 0$) and C$^-$ ($x < 0$), and the time spent, T_p^+ and T_p^-, in the corresponding state. Figure 4a shows the PDF of T_p for the stochastic process. For $T_p > 10$, the PDF of T_p is exponential, i.e., PDF of $T_p \sim \exp(sT_p)$. The various peaks at $T_p < 10$ correspond approximately to multiples of the resonant period (≈ 2.2) of the noise free system. When $f_o = 0.3$, the potential well is no longer symmetric and deepens for $x > 0$ (Figure 4a), with the effect of increasing the stability of the C$^+$ state relative to that of the C$^-$ state. As seen from Figure 4 changes in the PDF of T_p as f_o is varied are very similar to those in the Lorenz system as λ_r is varied (due to the imposed forcing). (Identical results were obtained when the linear term in $U(x)$ was replaced by a cubic term, showing that changes in $U(x)$ near $x = 0$ are not the cause of the changes in the slope when $f_o \neq 0$.) Here, as we vary f_o we change the depth of the wells, changing the degree of stability of the different regimes. The dominant effect in both cases is to change the frequency of occurrence of the extremely persistent events.

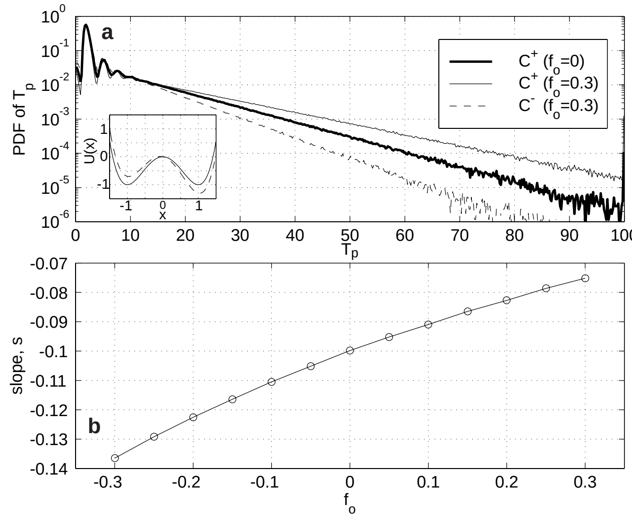

Figure 4. (a) PDF of T_p^+ and T_p^- for the stochastic process showing exponential behavior (ln PDF of $T_p \sim sT_p$). Inset: potential $U(x)$ for $f_o = 0$ (solid line) and $f_o = 0.3$ (dashed line). (b) Slope, s, of ln PDF of T_p^+, versus the stability parameter f_o. As f_o increases, the $x > 0$ potential well deepens, thus increasing the stability of the C$^+$ regime (s becomes less negative).

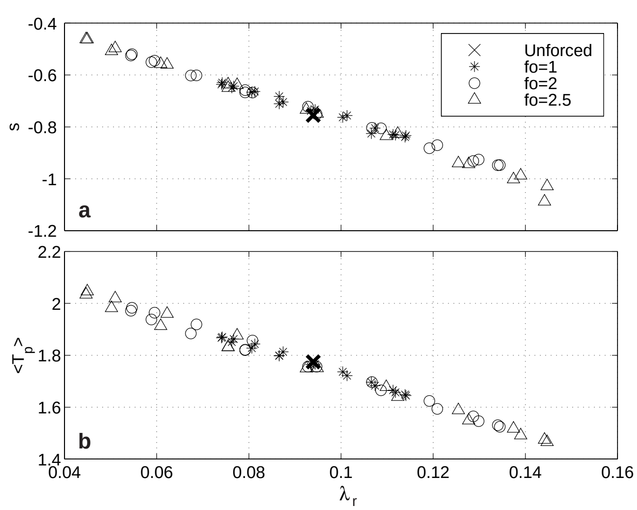

Figure 3. Plots of (a) slope, s, and (b) mean persistence time, $< T_p >$, versus λ_r for a range of forcing angles (θ) and amplitudes (f_o), showing the enhanced sensitivity of the frequency of occurrence of persistent events to the imposed forcing even as the mean changes by a small factor. Changes in λ_r are related to the forcing by $\Delta\lambda_r \approx f_o A \sin(\theta + \phi)$, where A ($= 0.02$) and ϕ ($-157.6°$) are constants which depend only on the parameters of the system (r,σ,b).

It is important to note that the slopes of the PDF's in both the Lorenz system and the stochastic system are largely insensitive to how we define the two regimes. For example, when we define C$^+$ and C$^-$ by $x > x_d$ and $x < x_d$, respectively, with $x_d \neq 0$, the slopes of the PDF's remain essentially unchanged. It is the stability of the regime, which is unaffected by exactly where we draw the boundary of the regime, that sets the persistence of the long–lived events in both systems.

4. Implications

Palmer [1999] had shown that small external forcing resulted in small alterations in the average time the Lorenz system spent in each of two regimes. Arguably, the more significant result is the dramatic change in the frequency of occurrence of extremely persistent events. Translating this result into climate terms, we might imagine that C$^-$ represents wet spells and C$^+$ dry spells, and the forcing is the impact of greenhouse gases on the climate system. An overall reduction in rainfall would be a problem in many areas, but an increase in prolonged periods of drought would be devastating. (According to USAID, the drought is the single greatest cause of human misery.) As a rule, both human systems and natural ecosystems have more difficulty adapting to extreme events than to changes in the mean.

We have further shown that this change in PDF of the persistence time in a regime is directly related to the stability of the regime – in fact, for the lorenz system, to the linear stability at the fixed point of the regime. A stochastically forced double well system displays similar sensitivity to regime stability. Seeing exponential changes in the occurrence of persistent events in both systems leads us to speculate that our results may be generally applicable to simple dynamical and stochastic systems. It would be interesting to look for similar extreme sensitivities of persistent events in more complicated systems. If it were true of general circulation models subject to enhanced greenhouse forcing, then we would expect more catastrophic human consequences from global warming than a simple rise in mean temperature.

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References

Corti, S., F. Molteni, and T. N. Palmer, Signature of recent climate change in frequencies of natural atmospheric circulation regimes, *Nature*, *398*, 799–802, 1999.

Gardiner, C. W., *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, Springer-Verlag, 1985.

IPCC, Climate Change 1995: The science of climate change, contribution of working group 1 to the second assesment report of the Intergovernmental Panel on Climate Change, Cambridge University Press, Cambridge, U.K., 572 pp., 1996.

Lorenz, E. N., Deterministic nonperiodic flow, *J. Atmos. Sci.*, *20*, 130–141, 1963.

Marshall, J., and F. Molteni, Towards a dynamical understanding of planetary-scale regimes, *J. Atmos. Sci.*, *50*, 1792–1818, 1993.

Palmer, T. N., A nonlinear dynamical perspective on climate prediction, *J. Climate*, *12*, 575–591, 1999.

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