Department of Applied Physics and Applied Mathematics
Columbia University
APPH E4210. Geophysical Fluid Dynamics
Spring 2004

Problem Set 10

(Due April 22, 2004)

1. In class we developed the shallow water quasigeostrophic equations with the assumption that the bottom topography h_b/H_o is O(Rossby number). If, instead, we assume that h_b/H_o is O(1), what does this imply about the O(1) geostrophic velocity?

2. (*Steady internal waves in a background flow.*) Consider an unbounded, stratified, incompressible fluid flowing over sinusoidally varying bottom topography. For simplicity, assume that the flow is two dimensional (in the x-z plane) and that rotation plays a negligible role.

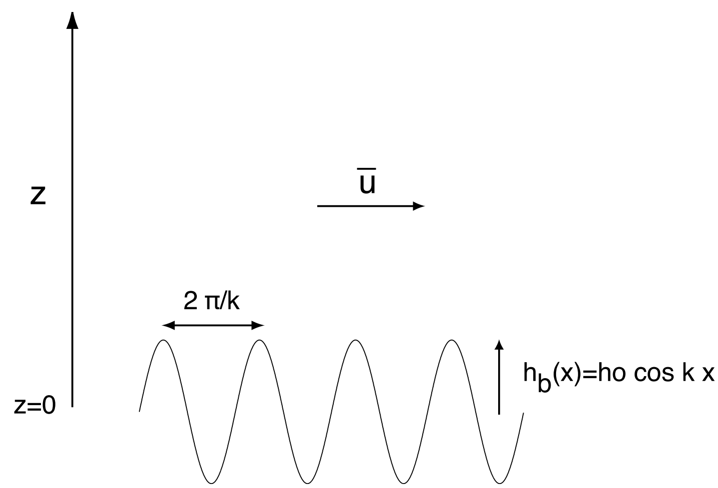

Figure 1: Definition sketch for problem 3.

(a) Write down the governing equations (momentum, continuity, density) for small perturbations about a steady zonal flow $\overline{u}(z)$.

(b) Derive a single governing equation for the relative vorticity of the disturbance. It is convenient to write this equation in terms of a streamfunction ψ for the disturbance field.

(c) Assuming that the disturbance is steady in time and periodic in x (or alternatively, decays as $x \to \pm\infty$), show that the vorticity equation is

$$\nabla^2\psi + \left(\frac{N^2}{\overline{u}^2} - \frac{\overline{u}_{zz}}{\overline{u}}\right)\psi = 0$$

(d) For topography of the form $h_b(x) = h_o \cos kx$, what is the linearized boundary condition at $z = 0$ (see sketch).

(e) Assuming constant N and $\overline{u}$, and $k^2 > N^2/\overline{u}$ (with $k > 0$), find the solution to the equation in part (c). Make a sketch of this solution, showing, in particular, the phase relation of the pressure field to the horizontal topography. Calculate the vertical flux of energy $\overline{p'w'}$ due to the wave. ($\overline{()}$ represents a horizontal average over a wavelength.) Also compute the force exerted by the fluid on the topography. This is the horizontal force (per unit area) exerted by the pressure field on the topography, and (in linearized form) is given by:

$$F = \frac{k}{2\pi} \int_0^{2\pi/k} p'(z=0)(dh/dx)dx$$

Give a physical explanation of this result. (Note that the topography exerts an equal and opposite force on the fluid. This force is known as the *drag*.)

(f) Repeat part (e), but now for the case $k^2 < N^2/\overline{u}$ ($k > 0$). Note that the general solution consists of two parts. To pick the correct solution, you must impose the additional constraint that the vertical energy flux be upward. This is called the *radiation condition*. Make a sketch of the solution, showing the pressure field and the phase lines of the wave. In what direction do the phase lines tilt with height? Show that the vertical energy flux is equal to the rate at which work is done by the fluid on the topography. Give a physical explanation of this result.

Note: $\overline{u}k$ is the frequency of encounter of the fluid particles with the crests of the topography. As the fluid travels over the bumps, internal gravity waves with this *intrinsic* frequency (i.e., frequency measured in a frame of reference moving with the background wind) are generated. (In a fixed reference frame the frequency is zero, since we are considering stationary waves.) When this frequency is greater than N (part e), the fluid cannot sustain internal gravity waves and the disturbance is *trapped* near the bottom. When it is less than N (part f), the fluid is able to sustain waves and energy can propagate vertically. These waves are known as *lee waves*.