Department of Applied Physics and Applied Mathematics
Columbia University
APPH E4210. Geophysical Fluid Dynamics
Spring 2004

Problem Set 5

(Due Feb 26, 2004)

1. *Dispersion relation.* Find the dispersion relation for the following equations (subscripts represent partial derivative, and ∇^2 is the 3-dimensional Laplacian):

   (a) $\psi_{tt} = c^2 \nabla^2 \psi$

   (b) $\psi_t = \nu \psi_{xx}$

   (c) $\psi_{tt} = c^2 \nabla^2 \psi + \alpha^2 \nabla^2 \psi_{tt}$. (Longitudinal waves in bars.)

   (d) $\psi_{tt} + \alpha^2 \psi_{xxxx} = 0$. (Beam equation.)

   (e) $\psi_{tt} = c^2(\psi_{xx} - \alpha \psi_{xxxx})$. (Piano string.)

   Give a physical interpretation of the dispersion relation you found in (b) above. (Did you notice anything peculiar about it?)

2. *Ray tracing*. In this problem you will work out the refraction of surface gravity waves as they approach the shore. Consider waves generated in deep water and moving onshore. Assume that the depth of the water $H(x)$ decreases linearly from 150 m in deep water to 130 m near the shore over a distance of 2 km. (Orient axes such that x (y) is perpendicular (parallel) to the coast.) The depth is constant in the longshore direction, i.e., the isobaths are parallel to the coast. For simplicity, assume that the shallow water dispersion relation for gravity waves is valid, i.e., $\omega = k\sqrt{gH(x)}$ in the "slowly varying" approximation, and that waves of only a single angular frequency, $\omega = 50$ radian/s, are generated. Set up, and solve, the ray theory equations for a wave vector $\mathbf{k} = (k, l)$. *Hints*: It is easier to pick an initial value for l (with 0 representing a vector perpendicular to the shore), and allow the x component to be determined by the dispersion relation. (Remember, we are interested in waves traveling *onshore*.) Plot ray paths for several different initial values of l. You may do the calculation analytically or numerically (in your favorite programming language). Superimpose on this plot (a sketch is acceptable), a few typical wave crests to show how the wavelength changes as the shore is approached? (Use the dispersion relation, and the fact that the wave crests are perpendicular to the rays.) If you are stymied, look at `http://www.ldeo.columbia.edu/~spk/Classes/APPH4210_GFD/raytracing_sound.m` for an example.

3. *Condition for linearization*. In class we claimed that linearization is valid when particle speeds are much less than the phase speed. Give a scaling argument to prove this statement.

4. For a plane wave solution, $\psi = A \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t)$, show that

$$< (\mathrm{Re}\, \psi)^2 >= \frac{1}{2} A A^*,$$

   where, $<>$ represents an average over a complete cycle of the phase, and A^* is the complex conjugate of A. We will frequently exploit this result.