Department of Applied Physics and Applied Mathematics
Columbia University

APPH E4210. Geophysical Fluid Dynamics
Spring 2004

Problem Set 6

(Due March 4, 2004)

1. *Warm up and review*. Consider a homogeneous layer of depth H. Assuming no rotation, show that the perturbation pressure p' satisfies Laplace's equation. Derive the linearized boundary conditions p' is subject to at the bottom, $z = -H$, and the top, $z = 0$. Solve the resulting problem, and find the gravity wave dispersion relation. (Do not make the hydrostatic approximation.) Assuming that the wave vector is directed in the x direction, i.e., $\mathbf{k} = (k, 0)$, make a plot of the ratio of group velocity to phase speed as a function of kH.

2. *Energetics of internal waves.* Consider an unbounded fluid on the f-plane, with constant buoyancy frequency N. Making the Boussinesq approximation, derive an equation governing the vertical velocity component w. Substitute a plane wave solution $w = W_o \exp i(kx + ly + mz - \omega t)$ to find the dispersion relation for internal gravity waves. Also find, in terms of W_o, the horizontal components of velocity (u,v), the perturbation pressure p', and the perturbation density ρ'. Use the governing equations to derive a conservation law for the energy density E (energy per unit volume). Write E as a sum of kinetic and potential energy terms. Finally, for a plane wave, find the ratio of *average* kinetic to potential energy, where the average is over a complete cycle of the phase. (You may find it useful to review problem 5 from last weeks' assignment.) When the ratio is 1, we say that the energy is *equipartitioned*.

3. *Normal modes for the ocean*. The separation of variables procedure we applied in class results in the following equation for the vertical structure function $\hat{h}$:

$$\frac{d^2\hat{h}}{dz^2} + \frac{N^2(z)}{c^2}\hat{h} = 0,$$

subject to the following (linearized) boundary conditions:

$$\hat{h}(z = -H) = 0 \ \text{ and } \ \hat{h}(z = 0) = 0.$$

(Here, we have made the Boussinesq and rigid lid approximations.) Given $N(z)$, these equations define a Sturm-Liouville eigen problem for the eigenfunctions $\hat{h}(z)$ and the eigenvalues $1/c^2$. Using an observed profile of N, calculate (and plot!) the first 5 (in order of decreasing values of c) eigenfunctions and the corresponding values of c in two different ways:

(a) Numerically, and

(b) Using the WKBJ approximation.

The N profile can be downloaded from CourseWorks or `http://www.ldeo.columbia.edu/~spk/Classes/APPH4210_GFD/N_profile`. (Assume $H = 4290$ m.) *Hints*: When solving the problem numerically, beware that this is a boundary value problem. (Canned routines are generally designed to solve initial value problems.) Also, the WKB part of the problem is not difficult at all. The solution we derived in class is applicable with only slight modification. The WKB solution for the "equivalent phase speed" c is very useful in practice.