Department of Applied Physics and Applied Mathematics
Columbia University
APPH E4210. Geophysical Fluid Dynamics
Spring 2004

Problem Set 7

(Due March 11, 2004)

1. Consider the shallow water equations on an f-plane, and a plane wave solution of the form $\eta = \mathrm{Re}\,\eta_o \exp i(kx + ly - \omega t)$.

   (a) Find the velocity field, (u,v), in terms of η.

   (b) Write the flow field in terms of a component parallel to the wave vector ($u_\parallel$) and a component perpendicular to the wave vector ($u_\perp$). Show that the horizontal velocity vector traces out an ellipse. In which direction (clockwise or counter clockwise) does the velocity vector rotate?

2. *Poincare Waves*. Show that the perturbation potential vorticity (PV), q', of a Poincare wave is exactly zero. What does this imply about the relation between the vorticity ξ and the surface elevation η? (What this shows is that these waves "carry" no PV, and that the final geostrophic steady state can be determined from the initial PV distribution.)

3. *Geostrophic adjustment*. Consider the shallow water equations on an f-plane. Suppose that at $t = 0$, the surface elevation and meridional velocity v are both zero and the zonal velocity is given by

$$u = U_o, \; -L \leq y \leq L,$$

   and zero elsewhere.

   (a) Write down the appropriate Klein-Gordon equation governing the time evolution of η.

   (b) Write the solution as the sum of a time-dependent homogeneous solution (η_h) and a steady particular solution (η_s). Find the steady, geostrophic solution η_s. *Hint*: You will find that the problem to be solved is a 2d order, inhomogeneous ODE, which requires the specification of 2 boundary conditions (or constraints). Apparently, the only boundary conditions available are that η not blow up as $y \rightarrow \pm\infty$. What to do? Recall that a similar situation is encountered when solving for the Green's function. (If this sounds unfamiliar, look it up in any standard ODE or PDE book, Haberman, say.) There, and here too, we *integrate* the differential equation over a small interval centered about some point y_o, and then let the interval go to zero. (The choice of y_o

depends on the problem at hand.) This establishes the continuity (or lack thereof) of η and $d\eta/dy$ across y_o. The change in η or its derivative across y_o is known as a "jump condition" and provides us with the necessary constraints. It is easier than it sounds.

(c) Use the momentum equations to find the geostrophic velocity field.

(d) Compute the ratio R of the total energy in the final geostrophic state to that in the initial state. Express, and make a plot of, this ratio as a function of $L/\lambda_{\rm d}$, where $\lambda_{\rm d}$ is the deformation radius.

(e) What are the initial conditions satisfied by $\eta_{\rm h}$? Solve the homogeneous problem subject to these initial conditions. You may do it numerically or analytically. (I have not attempted the latter, but would be very happy if one of you can do it.) Make plots of the full time dependent solution $\eta = \eta_{\rm h} + \eta_{\rm s}$ at several times showing the approach to a steady state. *Hint*: If you do the problem numerically, it may be easiest to write $\eta_{\rm h}$ as a superposition of plane wave solutions (making use of the dispersion relation) and then evaluate the Fourier integral using the discrete Fourier transform. (A few lines of code in Matlab!)