Department of Applied Physics and Applied Mathematics
Columbia University
APPH E4210. Geophysical Fluid Dynamics
Spring 2005

Problem Set 4

(Due Feb 24, 2005)

1. *Energetics of internal waves.* Consider an unbounded fluid on the f-plane, with constant buoyancy frequency N.

   (a) Write down the linearized governing equations in the Boussinesq approximation.

   (b) If the vertical velocity component w is given by a plane wave solution $w = W_o \exp i(kx + ly + mz - \omega t)$, find, in terms of W_o, the horizontal components of velocity (u,v), the perturbation pressure p', and the perturbation density ρ'.

   (c) Use the governing equations to derive a conservation law for the energy density E (energy per unit volume). Write E as a sum of kinetic and potential energy terms.

   (d) Finally, for a plane wave, find the ratio of *average* kinetic to potential energy, where the average is over a complete cycle of the phase. When this ratio is 1, we say that the energy is *equipartitioned*. *Hint*: For a plane wave solution, $\psi = A \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t)$, it can be shown that $< (\mathrm{Re}\,\psi)^2 >= \frac{1}{2} A A^*$, where $<>$ represents an average over a complete cycle of the phase, and A^* is the complex conjugate of A. This result proves to be rather convenient when computing averages of the kind required here.

2. *Normal modes for the ocean*. The separation of variables procedure we applied in class results in the following equation for the vertical structure function $\hat{h}$:

$$\frac{d^2 \hat{h}}{dz^2} + \frac{N^2(z)}{c^2} \hat{h} = 0,$$

   subject to the following (linearized) boundary conditions:

$$\hat{h}(z = -H) = 0 \ \text{ and } \ \hat{h}(z = 0) = 0.$$

   (Here, we have made the Boussinesq and rigid lid approximations.) Given $N(z)$, these equations define a Sturm-Liouville eigen problem for the eigenfunctions $\hat{h}(z)$ and the eigenvalues $1/c^2$. For reasons we will discuss in class, c is known as the “equivalent phase speed.” Using an observed profile of N from the Pacific Ocean, calculate (and plot!) the first 5 (in order of decreasing values of c) eigenfunctions and the corresponding values of c in two different ways:

(a) Numerically, and

(b) Using the WKBJ approximation.

Please provide a printout of the scripts or programs you use to solve and plot the solution. The N profile can be downloaded from CourseWorks. (Assume $H = 4290$ m.) *Hints*: When solving the problem numerically, beware that this is a boundary value problem. (Canned routines are generally designed to solve initial value problems.) In general, the numerical solution of eigenvalue problems is a nontrivial business, but for present purposes a simple minded approach in which the above ODE is discretized using finite differences to give a matrix eigen problem $\mathbf{Ax} = \lambda\mathbf{Bx}$ is more than adequate. (Note that this matrix equation defines a *generalized eigenvalue problem* which you can solve quite easily in software such as MATLAB.) Also, the WKB part of the problem is not difficult at all. The solution we derived in class is applicable with only slight modification. The WKB solution for the "equivalent phase speed" c is very useful in practice.