Department of Applied Physics and Applied Mathematics
Columbia University
APPH E4210. Geophysical Fluid Dynamics
Spring 2005

Problem Set 5

(Due March 10, 2005)

1. Consider the shallow water equations on an f-plane, and a plane wave solution of the form $\eta = \mathrm{Re}\, \eta_o \exp i(kx + ly - \omega t)$.

   (a) Find the velocity field, (u,v), in terms of η.

   (b) Write the flow field in terms of a component parallel to the wave vector ($u_{\parallel}$) and a component perpendicular to the wave vector ($u_{\perp}$). Show that the horizontal velocity vector traces out an ellipse. In which direction (clockwise or counter clockwise) does the velocity vector rotate?

2. *Geostrophic adjustment*. Consider the shallow water equations on an f-plane. Suppose that at $t = 0$, the velocity field is zero and the surface elevation is given by

$$\eta = \eta_o, \; -a \leq y \leq a,$$

   and zero elsewhere. (Make sure to attach plots of all solutions and printouts of any scripts/programs.)

   (a) Write down the appropriate Klein-Gordon equation governing the time evolution of η.

   (b) Write the solution as the sum of a time-dependent homogeneous solution (η_{h}) and a steady particular solution (η_{s}). Find the steady, geostrophic solution η_{s}. *Hint*: You will find that the problem to be solved is a 2d order, inhomogeneous ODE, which requires the specification of 2 boundary conditions (or constraints). Apparently, the only boundary conditions available are that η not blow up as $y \to \pm\infty$. What to do? Recall that a similar situation is encountered when solving for the Green's function. There, and here too, we *integrate* the differential equation over a small interval centered about some point y_o, and then let the interval go to zero. (The choice of y_o depends on the problem at hand.) This establishes the continuity (or lack thereof) of η and $d\eta/dy$ across y_o. The change in η or its derivative across y_o is known as a "jump condition" and provides us with the necessary constraints.

   (c) Use the momentum equations to find the geostrophic velocity field.

(d) Compute the ratio R of the total energy in the final geostrophic state to that in the initial state. Express, and make a plot of, this ratio as a function of $a/\lambda_{\rm d}$, where $\lambda_{\rm d}$ is the deformation radius.

(e) *Transient solution.* Now that you have found the steady (particular) solution, lets now calculate the time-dependent (homogeneous) solution. While this transient solution can be found analytically by means of Fourier or Laplace transforms, the inverse transforms are difficult to work out. (Of course, we could just look these up in tables!) Here, I walk you through the steps necessary to obtain the solution numerically. The basic idea is to use the discrete Fourier transform (implemented as the FFT in most math software including matlab) to do the inverse transform.

i. Write down the PDE for the homogeneous part, $\eta_{\rm h}$, and the initial conditions it is subject to.

ii. Take the Fourier transform of the equation (in the spatial direction) and solve the resulting ODE (for the Fourier transform). Equivalently, you could simply write down the solution as a sum of left- and right-going plane wave solutions using the known (Poincare) dispersion relation for this equation.

iii. Use the DFT (FFT) to invert the transform solution back into physical space.

iv. Make plots of the full time dependent solution $\eta = \eta_{\rm h} + \eta_{\rm s}$ at several times showing the approach to a steady state. If you nondimensionalize time and space appropriately, your life will be greatly simplified.