Department of Earth and Environmental Science
Columbia University

EESC G9810. Mathematical Earth Science Seminar: Vibrations and Waves

Spring 2003

Problem Set 3

(Due Feb 24, 2003)

1. We discussed Stokes' law in class for viscous damping. Consider a lead sphere of diameter D dropped in a container of glycerin at room temperature. As the ball falls through the fluid it accelerates under gravity but this increases the viscous drag. At some velocity the two forces balance. Compute this *terminal velocity* and the associated Reynolds number (UD/ν). Repeat the calculation for a sphere falling through air. Hint: You will need to look up various parameters in a handbook (the CRC Handbook of Physics and Chemistry is available online).

2. In class we derived an expression for the *mean* time and distance (also called the *mean free path*) between collisions of gas molecules. In general, however, each molecule spends a *different* amount of time and distance between collisions. We can thus regard the collision time and distance as *random variables*. Derive the *probability density functions* (PDFs) of these random variables. This is an important problem in physics and, more generally, the reasoning used can be applied to a wide range of problems (including radioactive decay). The key assumption is that the likelihood of a particle colliding with another is *independent* of how long it has been since the last collision. Thus the probability of a collision in a time interval Δt can be regarded as constant. Such a process is known as Markovian or memoryless. Here is a recipe for solving this problem:

   (a) Let $P(t)$ be the probability that a molecule survives a time t without suffering a collision.

   (b) Now let γdt be the probability that a molecule suffers a collision between time t and $t + dt$. γ is thus the probability per unit time that a molecule suffers a collision (the "collision rate"). Assume γ is constant.

   (c) Use the above to obtain an expression for $P(t + dt)$ and thus an ODE for $P(t)$.

   (d) Now compute $p(t)dt$, the probability that a molecule after surviving without collision for a time t suffers a collision in the time interval between t and $t + dt$. $p(t)$ is the PDF we desire.

   (e) Normalize $p(t)$ and compute its first moment. This is the average time between collisions. Estimate its value for the atmosphere at sea level.

(f) Apply the above reasoning to derive the PDF of the collision distance λ. Compute its first moment. This is the mean free path. Estimate its value for the atmosphere at a height of 100 km.

An alternative, and very instructive, way to solve this problem is to divide time into small intervals dt and make use of the binomial theorem. Taking the limit $dt \to 0$ will recover the result derived above. I leave this as an exercise (this is a problem set after all!).