Department of Earth and Environmental Science
Columbia University

EESC G9810. Mathematical Earth Science Seminar: Vibrations and Waves

Spring 2003

Problem Set 4

(Due Apr 28, 2003)

1. Problem 5-9 in French (normal modes of CO_2 molecule)

2. First do problem 5-9 in French. There is something not right with this simple model of the CO_2 molecule. The *observed* ratio of the two normal mode frequencies is 1.692 (not what you found). The problem is that if the center (C) atom is held fixed while the right (O) atom is moved, there is no force communicated to the left hand O atom. In reality, when the right hand oxygen atom is moved, the distribution of electrons around the carbon is changed and this causes a slight change in the bond strength between the carbon and the left hand oxygen. Thus, a more realistic model would be to couple the two oxygen atoms by a third spring (spring constant k'): O–k–C–k–O
$|---k'---|$
Find the normal mode pattern, as well as the frequencies and their ratios for this new system. Use the observed ratio to determine k'/k. Should k' be positive (repulsion) or negative (attraction)?

3. General problem of diagonalization for 2 coupled oscillators: $|–k_1–m_1–k_c–m_2–k_2–|$
   (a) Write down the equations of motion as (a) ODEs and, (b) in matrix form.
   (b) Write down the equations for the kinetic (T) and potential (V) energy.
   (c) Find the coordinate transformation which gives simplified, uncoupled T and V. (Recall the theorem we proved in class: this transformation also uncouples the equations of motion.) Do this in two steps:
   Step 1: Simplify T by defining a new set of coordinates $\mathbf{y} = \mathbf{M}^{1/2}\mathbf{x}$ where $\mathbf{x}$ is the vector of displacements and $\mathbf{M}$ the "mass" or "loading" matrix. Show schematically what happens to T when this transformation is applied. What is the condition for normal modes?
   Step 2: Find V in the new coordinates $\mathbf{y}$. Now define a set of "normal mode" coordinates $\mathbf{q}$ which simplifies the expression for $V(y_1, y_2)$. Show this schematically.
   Finally, put it all together, i.e., find the connection between the original coordinates $\mathbf{x}$ and the normal mode coordinates $\mathbf{q}$. Also find the normal modes and normal mode frequencies.

4. Consider a system of N coupled oscillators (N particles of equal mass m coupled through springs with spring constant k). Find the normal modes of this system with *periodic boundary conditions*. Note that the eigen frequencies (dispersion relation) is the same as we found in class with "rigid" BCs. Show that there are precisely N normal modes. Write down the general solution to the the initial value problem (see the MATLAB scripts for hints). Show that there are precisely $2N$ undetermined constants. How are these determined?

5. Derive the heat/diffusion equation in 3 dimensions. Use conservation of energy and Fourier's Law of heat conduction.

6. Solve the heat equation $u_t = \kappa u_{xx}$ on the domain $0 < x < L$ with initial condition $u(x, 0) = f(x)$ and no-flux boundary conditions at $x = 0$ and $x = L$ (i.e., $u_x = 0$ at $x = 0$ and $x = L$). How does this solution differ from what we derived in class with BCs $u(0, t) = u(L, t) = 0$?