Plasma Physics and Controlled Fusion. Vol. 33, No. 6, pp. 573 to 581, 1991
Printed in Great Britain.
0741-3335/91 $3.00+ .00

AVERAGE MAGNETIC SURFACES IN TOKAMAKS*

S. J. CAMARGO† and I. L. CALDAS
Instituto de Física, Universidade de São Paulo, C.P. 20.516, 01498 São Paulo, SP, Brazil

(*Received* 4 *September* 1990; *and in revised form* 19 *December* 1990)

Abstract—An average invariant which describes average magnetic surfaces for a system without symmetry is obtained. The system is a Tokamak toroidal equilibrium perturbed by resonant helical windings. The magnetic field is a superposition of the magnetic fields of the equilibrium and the helical windings, and the corresponding vector potential is determined. An average vector potential is defined to obtain the average invariant. Analysis of the average surfaces showed that the magnetic islands move towards the plasma centre and decrease in width as the pressure increases.

1. INTRODUCTION

THE INVESTIGATION of three-dimensional MHD equilibria is much more complicated than that of axisymmetric equilibria. Because of the lack of symmetry, no exact Grad–Shafranov-like equation can be derived (FREIDBERG, 1982), and the equilibrium investigations have relied on asymptotic studies or numerical computations. The problem, thus, is very hard and has been the object of great interest, see, for instance, MOROZOV and SOLOV'EV (1966), FREIDBERG (1982), REIMAN and BOOZER (1984), SHAFRANOV (1966) and CARY (1984b).

The equilibrium of a Tokamak can be improved using resonant helical windings with the same helicity as the magnetic field (The PULSATOR TEAM, 1985; ROBINSON, 1985). These helical windings, however, may destroy the magnetic surfaces. This happens, because when the magnetic field has a symmetry, there are magnetic surfaces that can confine the plasma and perturbing fields, that break the symmetry, can lead to large changes in the topological structure of these magnetic surfaces (GRAD, 1985).

The magnetic surfaces of a plasma are studied in a Tokamak, where the toroidal equilibrium is modified by resonant helical windings. As the amplitude of the current in the helical windings is much smaller than that of the plasma current, the helical windings are regarded as a perturbation of the equilibrium. The magnetic field is thus considered as a superposition of the magnetic fields of the equilibrium and the helical windings.

The field lines can be described by means of a variational principle (CARY and LITTLEJOHN, 1982) using the vector potential of the system. Applying Noether's theorem to the Lagrangian of the problem, one can conclude that, if all components of the vector potential are independent of one of the coordinates, the component of the vector potential corresponding to this coordinate is an invariant (CARY and LITTLEJOHN, 1982).

Our problem has no symmetry, since the equilibrium has a toroidal symmetry and

* This paper is an expanded version of material which originally was a contributed presentation at the 17th EPS Plasma Physics Division Conference, Amsterdam, The Netherlands, June 1990.

† Present address: Max-Planck-Institut für Plasmaphysik, Boltzmannstraße 2, D-8046 Garching bei München, F.R.G.

the windings depend on a helical variable. Then Noether's theorem cannot be applied directly because there is a dependence on all three coordinates.

An average vector potential is then defined, independent of the poloidal angle. Using a suitable coordinate system, an approximate invariant is obtained, corresponding to the poloidal component of the average vector potential. Using this invariant, the average surfaces can be determined, describing approximately the problem.

The method used to obtain the average surfaces is explained in Section 2 (CARY, 1984a). Its application requires that the vector potentials of the equilibrium (Section 3) and of the toroidal helical windings (Section 4) are calculated. In Section 5 the superposition of the helical windings on the equilibrium is analyzed and the conclusions are given is Section 6.

2. AVERAGING METHOD

As the current in the helical windings is much smaller than the plasma current, the vector potential of the system $\mathbf{A}$ is the superposition of the vector potential of the equilibrium $\mathbf{A}^0$ and that of the perturbation $\tilde{\mathbf{a}}$:

$$\mathbf{A}(\rho, \theta, \varphi) \simeq \mathbf{A}^0(\rho, \theta) + \tilde{\mathbf{a}}(\rho, \theta, \varphi), \tag{1}$$

where ρ, θ, φ are the local coordinates shown in Fig. 1.

The helical windings are described by the following equations:

$$u = m\theta - n\varphi = \text{constant} \tag{2}$$

$$\rho = b, \tag{3}$$

where m and n are the number of periods of the helical field in the toroidal and poloidal directions, respectively, and b is the minor radius of the Tokamak.

The average vector potential is defined as the average over the poloidal angle θ on a line where u is constant (CARY, 1984a):

$$\bar{A}_i(\rho, u) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, A_i\left(\rho, \theta, \varphi = \frac{m\theta - u}{n}\right). \tag{4}$$

The average vector potential is of the form:

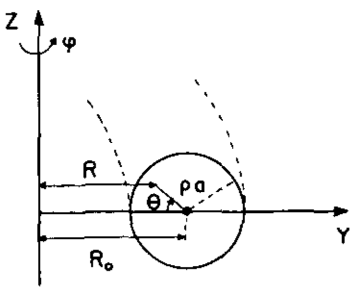

FIG. 1.—Coordinate system.

$$\bar{\mathbf{A}}(\rho,u) = \bar{A}_\rho(\rho,u)\hat{e}_\rho + \bar{A}_\theta(\rho,u)\hat{e}_\theta + \bar{A}_\varphi(\rho,u)\hat{e}_\varphi. \quad (5)$$

It is natural to use the variable u instead of φ. The average vector potential can then be written as (CARY, 1984a)

$$\bar{\mathbf{A}}(\rho,u) = \bar{A}_\rho(\rho,u)\hat{e}_\rho + \left(\bar{A}_\theta(\rho,u) - \frac{m}{n}\bar{A}_\varphi(\rho,u)\right)\hat{e}_\theta - \frac{1}{n}\bar{A}_\varphi(\rho,u)\hat{e}_u. \quad (6)$$

It is possible to describe the magnetic field lines through a variational principle in an arbitrary coordinate system (CARY and LITTLEJOHN, 1982):

$$\delta \int A_i(\mathbf{x})\frac{\mathrm{d}x_i}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = 0, \quad (7)$$

where λ is an arbitrary parameter and can be taken as one of the coordinates x_i. The cooresponding Lagrangian is then

$$L = A_i(\mathbf{x})\frac{\mathrm{d}x_i}{\mathrm{d}t}. \quad (8)$$

Application of Noether's theorem (CARY and LITTLEJOHN, 1982; HILL, 1951) to this Lagrangian can imply that one of the components is an invariant, if all the components of the vector potential are independent of the corresponding coordinate.

In the coordinate system considered in equation (6), the average vector potential does not depend on the poloidal angle θ. Therefore, in accordance with Noether's theorem (CARY and LITTLEJOHN, 1982), the component θ of the average vector potential is an invariant:

$$\Psi(\rho,u) = \bar{A}_\theta(\rho,u) - \frac{m}{n}\bar{A}_\varphi(\rho,u) = \text{constant}. \quad (9)$$

The theorem cannot be applied directly to the exact vector potential, as it depends on all three coordinates.

Although Ψ is an exact invariant, it is not the exact invariant of the problem. Following the same reasoning developed by CARY (1984a), for small values of ρ/R_0, the averaged vector potential is close to the actual vector potential. Therefore, the exact invariant of the approximate flow is an approximate invariant of the exact flow. The validity of the method derives from this fact.

3. TOROIDAL EQUILIBRIUM

Shafranov analyzed a plasma confined in a toroidal apparatus, using toroidal coordinates (SHAFRANOV, 1960). It is possible to transform these coordinates to local coordinates; it must be mentioned, however, that this transformation is not valid near the magnetic axis. The invariant which describes the equilibrium is (CAMARGO, 1989)

$$\psi^0 = \frac{\mu_0 I_p R_0}{4\pi}\left(1-\frac{\rho^2}{a^2}\right)\left(1-\frac{\rho}{R_0}(\Lambda+1)\cos\theta\right), \tag{10}$$

where the major and minor radii of the plasma are R_0 and a, respectively (see Fig. 1), I_p is the plasma current, μ_0 is the magnetic susceptibility and Λ is defined by (LA HAYE *et al.*, 1981)

$$\Lambda = \beta_p + \frac{l_i}{2} - 1. \tag{11}$$

In equation (11), β_p is the ratio of the kinetic pressure to the magnetic poloidal pressure of the plasma and l_i its internal inductance.

Figure 2 shows the magnetic surfaces described by curves of constant ψ^0, on the basis of the parameters of the TBR-1 Tokamak (VANNUCCI *et al.*, 1988; KUCINSKI *et al.*, 1990):

- $a = 8$ cm;
- $R_0 = 30$ cm;
- $I_p = 18$ kA;
- $\Lambda = 0.28$.

The function I, related to the invariant ψ^0, is

$$I^2 = I_e^2 + 2\frac{R_0^2}{a^2} I_p^2\left(\tfrac{5}{4}-\Lambda\right)\left(1-\frac{\rho^2}{a^2}\right)\left(1-\frac{\rho}{R_0}(\Lambda+1)\cos\theta\right), \tag{12}$$

where I_e is the current in the Tokamak coils, with a typical value of 600 kA.

The magnetic field can be obtained from the expression

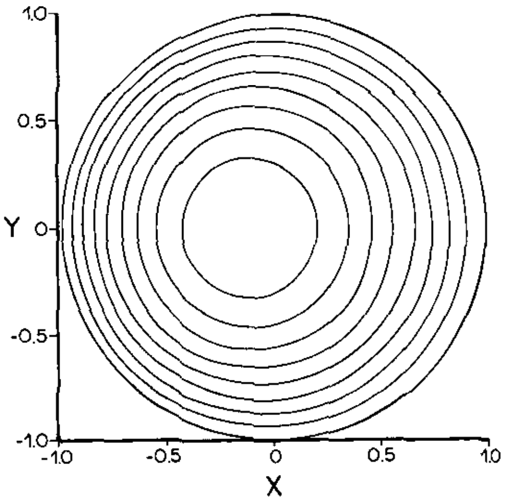

FIG. 2.—Surfaces of constant ψ^0 for $\Lambda = 0.28$, major Tokamak radius $R_0 = 30$ cm, minor Tokamak radius $a = 8$ cm. The scales are normalized to the minor Tokamak radius.

$$\mathbf{B} = \frac{1}{R}\left(\nabla\psi^0 \times \hat{e}_\varphi + \frac{\mu_0 I(\psi^0)}{2\pi}\hat{e}_\varphi\right), \tag{13}$$

where R is given by (Fig. 1)

$$R = R_0 - \rho\cos\theta. \tag{14}$$

The vector potential can be calculated from the following expression (BOOZER, 1986):

$$\mathbf{A} = \chi\nabla\theta - \Gamma\nabla\varphi, \tag{15}$$

where χ is the toroidal flux of the magnetic field,

$$\chi = \frac{1}{2\pi}\int \mathbf{B}\cdot \mathrm{d}\mathbf{S}_\mathrm{t}, \tag{16}$$

and Γ the poloidal flux of the magnetic field,

$$\Gamma = \frac{1}{2\pi}\int \mathbf{B}\cdot \mathrm{d}\mathbf{S}_\mathrm{p}. \tag{17}$$

Using expressions (10), (12), (13), (15), (16) and (17), it is possible to calculate the vector potential of the equilibrium, making some approximations and keeping terms to the order of $(\rho/R_0)^2$. Averaging the expression obtained, employing definition (4), the average vector potential is finally found to be:

$$\bar{A}^0_\rho = 0, \tag{18}$$

$$\bar{A}^0_\theta = \frac{\mu_0}{4\pi}\left(C\left(\frac{\rho}{2} + \frac{\rho^3}{8R_0^2} + \frac{\rho^5}{16R_0^4}\right) - \frac{D}{8C}\frac{\rho^3}{a^2} - \frac{D}{2C}(\Lambda+1)\left(\frac{\rho^3}{8R_0^2} + \frac{\rho^5}{12a^2R_0^2}\right)\right), \tag{19}$$

$$\bar{A}^0_\varphi = \frac{\mu_0 I_\mathrm{p}}{4\pi}\left(1 - \frac{\rho^2}{a^2}\right)\left(1 - \Lambda\frac{\rho^2}{R_0^2}\right), \tag{20}$$

where

$$D = \frac{2I_\mathrm{p}^2}{a^2}\left(\tfrac{5}{4} - \Lambda\right) \tag{21}$$

and

$$C = \sqrt{D + \frac{I_\mathrm{c}^2}{R_0^2}}. \tag{22}$$

4. TOROIDAL HELICAL WINDINGS

A number of equidistant thin conductors wound on a circular torus carrying currents I_h in alternating directions is considered. The torus has a minor radius b and the toroidal helical windings are characterized by the number of periods m and n of the helical field in the poloidal and toroidal directions, respectively.

The scalar potential for this system is known (KUCINSKI and CALDAS, 1987):

$$\phi \simeq (-1)^{Nm+1}\left(1+\frac{\rho}{2R_0}\cos\theta\right)\frac{m\mu_0 I_h}{\pi N}\left(\left(\frac{\rho}{b}\right)^{Nm}\sin N(m\theta-n\varphi)\right.$$
$$-\frac{b}{4R_0}\left(\left(\frac{\rho}{b}\right)^{Nm+1}\frac{Nm+2}{Nm+1}\sin((Nm-1)\theta-Nn\varphi)\right.$$
$$\left.\left.+\left(\frac{\rho}{b}\right)^{Nm-1}\frac{Nm}{Nm+1}\sin((Nm+1)\theta-Nm\varphi)\right)\right) \quad (23)$$

where N is the harmonic considered and m the number of current pairs. Each term of the scalar potential corresponds to a certain resonance. The first term relates to the resonance m/n and the other two to the secondary resonances $(m-1)/n$ and $(m+1)/n$. Only the most important resonance is considered, since near the rational surfaces the contribution of the other rational surfaces is negligible. The expression taken for the scalar potential is then

$$\phi \simeq (-1)^{Nm+1}\left(1+\frac{\rho}{2R_0}\cos\theta\right)\frac{m\mu_0 I_h}{\pi N}\left(\frac{\rho}{b}\right)^{Nm}\sin N(m\theta-n\varphi). \quad (24)$$

The magnetic field can be obtained by means of

$$\tilde{\mathbf{b}} = \nabla\phi. \quad (25)$$

The same procedure adopted for the equilibrium is used to obtain the vector potential by means of the magnetic field fluxes. The vector potential is then calculated from equations (15), (16), (17), (24) and (25). Using the definition of the average vector potential (4), one obtains

$$\tilde{\bar{a}}_\rho = 0, \quad (26)$$

$$\tilde{\bar{a}}_\theta = (-1)^{Nm}\frac{\mu_0 I_h n}{2\pi^2 N(Nm+2)}\left(\frac{\rho}{b}\right)^{Nm}\frac{\rho}{R_0}\sin N(m\theta-n\varphi), \quad (27)$$

$$\tilde{\bar{a}}_\varphi = (-1)^{Nm}\frac{\mu_0 I_h m}{\pi N}\left(\frac{\rho}{b}\right)^{Nm}\cos N(m\theta-n\varphi). \quad (28)$$

5. AVERAGE MAGNETIC SURFACES

As the superposition of the vector potential is of concern here, the average vector potential is now known from equations (1), (18), (19), (20), (26), (27) and (28). An approximate invariant (9) can therefore be obtained by the method described in Section 1 (CARY, 1984a):

$$\Psi = \frac{\mu_0}{4\pi}\left(C\left(\frac{\rho}{2}+\frac{\rho^3}{8R_0^2}+\frac{\rho^5}{16R_0^4}\right)-\frac{D}{C}\frac{\rho^3}{8a^2}\right.$$
$$-\frac{D}{2C}(\Lambda+1)\left(\frac{\rho^3}{8R_0^2}+\frac{\rho^5}{12a^2R_0^2}\right)+\frac{m}{n}I_p\left(1-\frac{\rho^2}{a^2}\right)\left(1-\Lambda\frac{\rho^2}{2R_0^2}\right)\Bigg)$$
$$+(-1)^{Nm}\frac{\mu_0 I_h}{\pi N}\left(\frac{\rho}{b}\right)^{Nm}\left(\frac{n}{2\pi^2(Nm+2)}\frac{\rho}{R_0}\sin Nu+\frac{m^2}{n}\cos Nu\right), \quad (29)$$

where u was defined in equation (2).

Two values of Λ are considered, the typical value of the TBR-1 Tokamak ($\Lambda = 0.28$) and the limit of zero ratio of the kinetic to the magnetic pressures of the plasma ($\Lambda = -1$), which also corresponds to $l_i = 0$. This last case is studied in order to evaluate the effect of the pressure on the position and the width of the magnetic islands.

Figures 3 and 4 show curves of constant Ψ using the parameters of TBR-1 and $m = 3$, $n = 1$, $N = 1$. In Fig. 3 we use $\Lambda = 0.28$ and in Fig. 4 we employ the limit $\Lambda = -1$.

From the figures, it can be seen that in the limit of zero pressures, the average magnetic islands are bigger and nearer the plasma boundary. The effect of the pressure on the island width can be verified by calculating the island width using usual expressions.

The changing of the pressure changes the basic equilibrium, as ψ^0 and I depend on Λ, and therefore on the pressure. This causes a change in the q-profile and thus of the location and size of the islands.

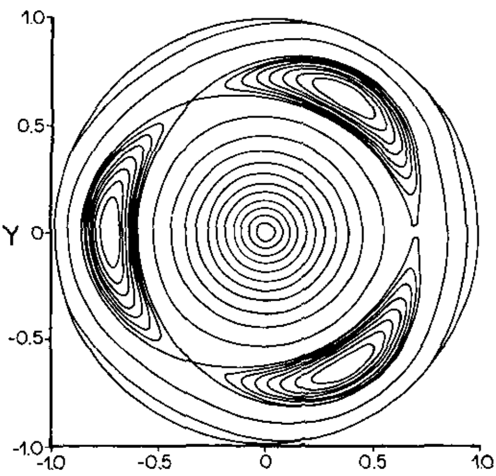

FIG. 3.—Surfaces of constant Ψ for $m = 3$, $n = 1$, $N = 1$, $I_p = 18$ kA, $I_e = 600$ kA, $I_h = 100$ A, $a = 8$ cm, $R_0 = 30$ cm, $b = 11$ cm and $\Lambda = 0.28$.

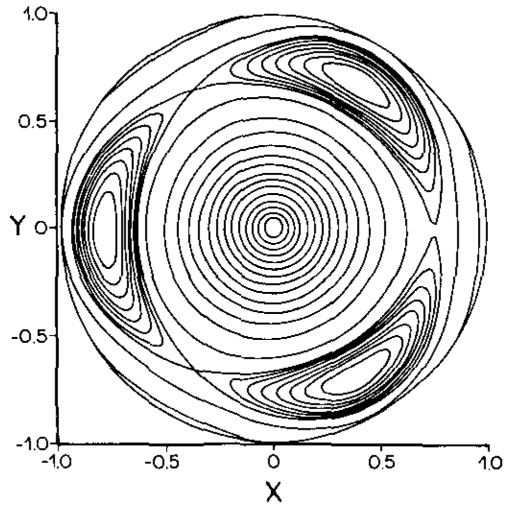

FIG. 4.—Surfaces of constant Ψ for $m = 3$, $n = 1$, $N = 1$, $I_p = 18$ kA, $I_c = 600$ kA, $I_h = 100$ A, $a = 8$ cm, $R_0 = 30$ cm, $b = 11$ cm and $\Lambda = -1$.

It should be stated that considering the pressure of the plasma is equivalent to a displacement of the magnetic axis, because when the pressure is not considered, the magnetic axis coincides with the geometric axis. The effect of the island width has already been observed by considering the displacement of the magnetic axis analytically and numerically (ZHENG and WOOTON, 1987).

6. CONCLUSIONS

The magnetic surfaces of a Tokamak perturbed by resonant helical windings were analyzed, although the system has no symmetry. An average invariant which approximately describes the problem was obtained. The average magnetic surfaces which form average magnetic islands were studied using the parameters of the TBR-1 Tokamak (CAMARGO and CALDAS, 1990).

It was found that the positions and widths of the islands depend on the plasma pressure. As the pressure increases, the islands become smaller and shift to the centre of the plasma.

The same analysis was made using an expression for the helical windings without the toroidal effect (CAMARGO, 1989). It was concluded that the toroidal effect results in magnetic islands which are slightly smaller than that obtained from the cylindrical approximation (CAMARGO, 1989).

Acknowledgements—The authors would like to thank W. P. DE SÁ (São Paulo University) for computational support and Dr D. CORREA-RESTREPO (Max-Planck-Institut für Plasmaphysik) for useful suggestions regarding this paper. This work was partially supported by FAPESP and CNPq.

REFERENCES

BOOZER A. H. (1986) *Physics Fluids* **29**, 4123.
CAMARGO S. J. (1989) M.Sc. Dissertation (in Portuguese), São Paulo University.
CAMARGO S. J. and CALDAS I. L. (1990) *Proc. 17th EPS European Conf. on Plasma Physics and Controlled Fusion* (Amsterdam, 1990), Vol. 14B, Part II, p. 675. European Physical Society.

CARY J. R. (1984a) *Physics Fluids* **27**, 119.
CARY J. R. (1984b) *Proc. Int. Conf. on Plasma Physics* (Lausanne, 1984), Vol. 1, p. 339.
CARY J. R. and LITTLEJOHN R. G. (1983) *Ann. Phys.* **151**, 1.
FREIDBERG J. P. (1982) *Rev. Mod. Phys* **54**, 801.
GRAD H. (1985) *Int. J. Fusion Energy* **3**, 33.
HILL E. L. (1951) *Rev. Mod. Phys* **23**, 253.
KUCINSKI M. Y. and CALDAS I. L. (1987) *Z. Naturforsch.* **42a**, 1124.
KUCINSKI M. Y., CALDAS I. L., MONTEIRO L. H. A. and OKANO V. (1990) *J. Plasma Physics* **44**, 303.
LA HAYE R. J., YAMAGISHI T., CHU M. S., SCHAFFER M. S. and BARD W. D. (1981) *Nucl. Fusion* **21**, 1235.
MOROZOV A. I. and SOLOV'EV S. L. (1966) in *Reviews of Plasma Physics* (Edited by M. A. LEONTOVICH), Vol. 2, p. 1. Consultants Bureau, New York.
THE PULSATOR TEAM (1985) *Nucl. Fusion* **25**, 1059.
REIMAN A. H. and BOOZER A. H. (1984) *Physics Fluids* **27**, 2446.
ROBINSON D. C. (1985) *Nucl. Fusion* **25**, 1101.
SHAFRANOV V. D. (1960) *Sov. Phys. JETP* **37**, 775.
SHAFRANOV V. D. (1966) in *Reviews of Plasma Physics* (Edited by M. A. LEONTOVICH), Vol. 2, p. 103. Consultants Bureau, New York.
VANNUCCI A., NASCIMENTO I. C. and CALDAS I. L. (1988) *Nuovo Cim.* D **10**, 1193.
ZHENG S. B. and WOOTON A. J. (1987) Report Tx 78712, Texas University.