Self-Similar Statistics in MHD Turbulence

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The fully developed decaying turbulence of 3-d resistive, viscous, incompressible magnetohydrodynamics is investigated using Elsasser variables and Hopf equation for probability distributions. The method is an extension of a previous work for Navier Stokes equations done by Foias et al. based on a suggestion by Hopf. It uses essentially self-similar properties of the statistics, which "almost" determine the turbulence spectrum up to a mild assumption on an unknown function. This spectrum is the well known Kolmogorov spectrum.

1. Introduction

Fully developed 3-d turbulence in Navier-Stokes fluids at high Reynolds numbers has a long history (see for example [1]). One of the milestones is the Kolmogorov spectrum [2], which was proposed in 1941 mainly on dimensional arguments. Hopf [3] introduced a general functional equation for the statistical description of turbulence and was the first author to sketch a "rigorous derivation" program of the Kolmogorov spectrum based on a two-parameter group of similarity transformations (see [3], page 120). An explicit accomplishment of this program, with proofs, can be found in [4] and [5], where from the self-similarity of the Hopf probability measure scaling properties of correlation functions and the energy spectrum are derived. Addition of appropriate "mild" physical assumptions to the mathematical scaling relationships leads to the well known $E_k = C\varepsilon^{2/3}k^{-5/3}$ spectrum of Kolmogorov. This kind of "rigorous derivation" does not close the topic because, for instance, observed intermittency even at rather high Reynolds numbers (see for example [6]) could escape to a "smooth" statistical treatment.

In this paper, we extend the derivation mentioned above to resistive viscous incompressible magnetohydrodynamics (MHD). It has been suggested a long time ago by Elsasser [7] to use variables bearing his name in a general context to describe MHD. This will be the first time, however, at least to our knowledge, that these variables are introduced together with an exact statistical derivation of the energy spectrum of 3-d MHD. As shown in this paper, the energy spectrum turns out to be essentially the Kolmogorov spectrum, as for the Navier-Stokes equation, up to an appropriate change of definition of E_k and ε. The missing rigor in the cited papers and in ours is the proof of global existence and uniqueness theorems for 3-d solutions of Navier-Stokes and MHD equations. For an account of that problem see [8].

The paper is organized as follows. In Section 2 the MHD system and its energy equation are given. The Hopf-Liouville equation and the averages are described in Section 3. We choose the probability distribution families and define the scale transformation in Section 4. In Section 5 we obtain the transformed MHD system. The correlation function and the energy spectrum are calculated in Section 6. The conclusion and the appendix are in Sections 7 and 8, respectively.

2. MHD System and Its Energy Equation

The MHD equations used in this paper describe a resistive, viscous, incompressible magnetofluid of unit density. They are given by

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} = -\nabla p + \frac{1}{\mu_M}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \nu\nabla^2\boldsymbol{v}, \tag{1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) + \eta\nabla^2\boldsymbol{B}, \tag{2}$$

where η, ν and μ_M are the magnetic diffusivity, the kinematic viscosity and the magnetic susceptibility. As usual, $\boldsymbol{v}$ is the velocity of the fluid, $\boldsymbol{B}$ the magnetic field and p the pressure. As the fluid is incompressible,

$$\nabla\cdot\boldsymbol{v} = 0, \tag{3}$$

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as well as

$$\nabla \cdot \boldsymbol{B} = 0\,. \tag{4}$$

Using vector formulas and the Elsasser variables [7]

$$\boldsymbol{P} = \boldsymbol{v} + \frac{1}{\sqrt{\mu_M}}\boldsymbol{B}, \tag{5}$$

$$\boldsymbol{Q} = \boldsymbol{v} - \frac{1}{\sqrt{\mu_M}}\boldsymbol{B}, \tag{6}$$

the MHD equations can be rewritten in the following symmetric way:

$$\frac{\partial \boldsymbol{P}}{\partial t} + (\boldsymbol{Q}\cdot\nabla)\,\boldsymbol{P} = -\nabla q + \nu_1 \nabla^2 \boldsymbol{P} \pm |\nu_2|\,\nabla^2 \boldsymbol{Q}, \tag{7}$$

$$\frac{\partial \boldsymbol{Q}}{\partial t} + (\boldsymbol{P}\cdot\nabla)\,\boldsymbol{Q} = -\nabla q \pm |\nu_2|\,\nabla^2 \boldsymbol{P} + \nu_1 \nabla^2 \boldsymbol{Q}, \tag{8}$$

which is very closely analogous to the Navier-Stokes equation. In (7) and (8)

$$\nu_1 = \tfrac{1}{2}(\nu+\eta), \tag{9}$$

$$\nu_2 = \tfrac{1}{2}(\nu-\eta) \tag{10}$$

and

$$q = p + \frac{B^2}{2\mu_M}. \tag{11}$$

The plus and the minus signs correspond respectively to $\nu > \eta$ and $\nu < \eta$.

The energy of the system is

$$E = \int \left(\frac{v^2}{2} + \frac{B^2}{2\mu_M}\right) \mathrm{d}\boldsymbol{x}, \tag{12}$$

where the volume of integration is the whole space.

The energy time-dependence can be found employing (1) and (2) and assuming appropriate boundary conditions at infinities, so that

$$\frac{\partial}{\partial t}\int \left(\frac{v^2}{2} + \frac{B^2}{2\mu_M}\right) \mathrm{d}\boldsymbol{x} = -\int \left\{\nu(\nabla\times\boldsymbol{v})^2 + \frac{\eta}{\mu_M}(\nabla\times\boldsymbol{B})^2\right\} \mathrm{d}\boldsymbol{x}, \tag{13}$$

which can also be written using Elsasser variables:

$$\frac{\partial}{\partial t}\int (P^2+Q^2)\,\mathrm{d}\boldsymbol{x} = -\int \{\nu\{\nabla\times(\boldsymbol{P}+\boldsymbol{Q})\}^2 + \eta\{\nabla\times(\boldsymbol{P}-\boldsymbol{Q})\}^2\}\,\mathrm{d}\boldsymbol{x}. \tag{14}$$

3. Hopf-Liouville Equation and Averages

The probability distribution function of the system (7) and (8), μ_t, is defined as a homogeneous statistical solution of the Hopf-Liouville equation as given in [5]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \Phi(\boldsymbol{P},\boldsymbol{Q})\,\mathrm{d}\mu_t(\boldsymbol{P},\boldsymbol{Q}) = -\int \mathrm{d}\mu_t \{\nu_1((\boldsymbol{P},\Phi'_P)) \pm |\nu_2|\,((\boldsymbol{Q},\Phi'_P)) + (\{\boldsymbol{Q}\cdot\nabla\}\,\boldsymbol{P},\Phi'_P) \pm |\nu_2|\,((\boldsymbol{P},\Phi'_Q)) + \nu_1((\boldsymbol{Q},\Phi'_Q)) + (\{\boldsymbol{P}\cdot\nabla\}\,\boldsymbol{Q},\Phi'_Q)\}, \tag{15}$$

where

$$(\boldsymbol{a},\boldsymbol{b}) = \int \boldsymbol{a}\cdot\boldsymbol{b}\,\mathrm{d}\boldsymbol{x}, \tag{16}$$

$$((\boldsymbol{a},\boldsymbol{b})) = \sum_{i=1}^{3}\int \nabla a_i \cdot \nabla b_i\,\mathrm{d}\boldsymbol{x}, \tag{17}$$

and Φ'_P is the functional derivative of the functional Φ with respect to $\boldsymbol{P}$.

In some volume V we introduce:

$$|\boldsymbol{P}|_V^2 \equiv \frac{1}{|V|}\int_V |\boldsymbol{P}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} \tag{18}$$

and

$$\|\boldsymbol{P}\|_V^2 \equiv \frac{1}{|V|}\int_V |\nabla\boldsymbol{P}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} = \frac{1}{|V|}\int_V |\nabla\times\boldsymbol{P}|^2\,\mathrm{d}\boldsymbol{x}, \tag{19}$$

where the last equality is due to the fact that both $\boldsymbol{v}$ and $\boldsymbol{B}$ are divergence free fields and thus also $\boldsymbol{P}$ and $\boldsymbol{Q}$.

The ensemble averaged energy is then

$$e(\mu_t) = \int \{|\boldsymbol{P}|_V^2 + |\boldsymbol{Q}|_V^2\}\,\mathrm{d}\mu_t\,. \tag{20}$$

In the same way, the ensemble averaged energy dissipation terms can be written as

$$E_1(\mu_t) = \int \|\boldsymbol{P}+\boldsymbol{Q}\|_V^2\,\mathrm{d}\mu_t\,, \tag{21}$$

$$E_2(\mu_t) = \int \|\boldsymbol{P}-\boldsymbol{Q}\|_V^2\,\mathrm{d}\mu_t\,. \tag{22}$$

The associated energy equation is

$$\frac{\partial e(\mu_t)}{\partial t} + \nu E_1(\mu_t) + \eta E_2(\mu_t) = 0\,. \tag{23}$$

Defining the dissipation rate ε as

$$\varepsilon(t) = \nu E_1(\mu_t) + \eta E_2(\mu_t)\,, \tag{24}$$

the energy equation (23) is

$$\frac{\partial e(\mu_t)}{\partial t} + \varepsilon = 0\,. \tag{25}$$

This equation tells us that the flow will be decaying as well as the turbulence we are studying.

4. Choice of Probability Distributions and Scale Transformations

Following Hopf's idea [3] to choose of all possible probability distributions the ones that depend only on the parameters ν_1, $|\nu_2|$, $\varepsilon(t)$, we define the family

$$\mu_t = \mu^{\nu_1, |\nu_2|, \varepsilon(t)}. \tag{26}$$

Similarly to [3, 4, 5], we demand this family to be preserved by change of scales of the MHD variables given by

$$\sigma_{\xi,\lambda}\, \boldsymbol{P}(\boldsymbol{x}) = \xi\, \boldsymbol{P}(\lambda \boldsymbol{x}), \tag{27}$$

$$\sigma_{\xi,\lambda}\, \boldsymbol{Q}(\boldsymbol{x}) = \xi\, \boldsymbol{Q}(\lambda \boldsymbol{x}), \tag{28}$$

This means that

$$\int \phi(\sigma_{\xi,\lambda}\, \boldsymbol{P}, \sigma_{\xi,\lambda}\, \boldsymbol{Q})\, \mathrm{d}\mu_t(\boldsymbol{P}, \boldsymbol{Q}) \equiv \int \phi(\boldsymbol{P}, \boldsymbol{Q})\, \mathrm{d}\{\sigma_{\xi,\lambda}\, \mu_t(\boldsymbol{P}, \boldsymbol{Q})\}. \tag{29}$$

Accordingly, the average energy transforms with $\sigma_{\xi,\lambda}$ in the following way:

$$e(\sigma_{\xi,\lambda}\, \mu_t) = \int \{|\boldsymbol{P}|_V^2 + |\boldsymbol{Q}|_V^2\}\, \mathrm{d}(\sigma_{\xi,\lambda}\, \mu_t) = \xi^2 \int \{|\boldsymbol{P}|_{\lambda^3 V}^2 + |\boldsymbol{Q}|_{\lambda^3 V}^2\}\, \mathrm{d}\mu_t. \tag{30}$$

So

$$e(\sigma_{\xi,\lambda}\, \mu_t) = \xi^2\, e(\mu_t). \tag{31}$$

Similarly, the average dissipation terms transform as

$$E_1(\sigma_{\xi,\lambda}\, \mu_t) = \int \|\boldsymbol{P} + \boldsymbol{Q}\|_V^2\, \mathrm{d}(\sigma_{\xi,\lambda}\, \mu_t) = \xi^2 \lambda^2 \int \|\boldsymbol{P} + \boldsymbol{Q}\|_{\lambda^3 V}^2\, \mathrm{d}\mu_t. \tag{32}$$

Then

$$E_1(\sigma_{\xi,\lambda}\, \mu_t) = \xi^2 \lambda^2\, E_1(\mu_t) \tag{33}$$

and

$$E_2(\sigma_{\xi,\lambda}\, \mu_t) = \xi^2 \lambda^2\, E_2(\mu_t). \tag{34}$$

Equations (15) and (29) lead to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \Phi(\boldsymbol{P}, \boldsymbol{Q})\, \mathrm{d}(\sigma_{\xi,\lambda}\, \mu_t) = \frac{\mathrm{d}}{\mathrm{d}t} \int \Phi(\sigma_{\xi,\lambda}\, \boldsymbol{P}, \sigma_{\xi,\lambda}\, \boldsymbol{Q})\, \mathrm{d}\mu_t$$

$$= -\frac{1}{\xi\lambda} \int \mathrm{d}(\sigma_{\xi,\lambda}\, \mu_t) \left\{ \frac{\nu_1 \xi}{\lambda} ((\boldsymbol{P}, \Phi'_P)) \pm \frac{|\nu_2|\, \xi}{\lambda} ((\boldsymbol{Q}, \Phi'_P)) + (\{\boldsymbol{Q} \cdot \nabla\}\, \boldsymbol{P}, \Phi'_P) \pm \frac{|\nu_2|\, \xi}{\lambda} ((\boldsymbol{P}, \Phi'_Q)) + \frac{\nu_1 \xi}{\lambda} ((\boldsymbol{Q}, \Phi'_Q)) + (\{\boldsymbol{P} \cdot \nabla\}\, \boldsymbol{Q}, \Phi'_Q) \right\}. \tag{35}$$

It can be noticed then, that for $\tilde{\mu}_t$, defined as

$$\tilde{\mu}_t \equiv \sigma_{\xi,\lambda}\, \mu_t, \tag{36}$$

transformed "viscosities" can be introduced:

$$\tilde{\nu}_1 = \frac{\nu_1 \xi}{\lambda}, \tag{37}$$

$$|\tilde{\nu}_2| = \frac{|\nu_2|\, \xi}{\lambda}, \tag{38}$$

as well as a new energy rate $\tilde{\varepsilon}$:

$$\tilde{\varepsilon}(t) \equiv \tilde{\nu} E_1(\tilde{\mu}_t) + \tilde{\eta} E_2(\tilde{\mu}_t), \tag{39}$$

so that

$$\tilde{\varepsilon}(t) = \xi^3 \lambda\, \varepsilon(t). \tag{40}$$

Then

$$\sigma_{\xi,\lambda}(\mu^{\nu_1, |\nu_2|, \varepsilon}) = \mu^{\frac{\nu_1 \xi}{\lambda}, \frac{|\nu_2|\, \xi}{\lambda}, \xi^3 \lambda \varepsilon}. \tag{41}$$

To fix a reference distribution function we choose

$$\frac{(\nu_1 |\nu_2|)^{1/2}\, \xi}{\lambda} = 1, \tag{42}$$

$$\varepsilon \lambda\, \xi^3 = 1. \tag{43}$$

Then (31) can be written as

$$e(\mu^{\nu_1, |\nu_2|, \varepsilon}) = (\nu_1 |\nu_2|)^{1/4}\, \varepsilon^{1/2}\, \gamma, \tag{44}$$

where

$$\gamma \equiv e \left\{ \mu^{\left(\frac{\nu_1}{|\nu_2|}\right)^{1/2}, \left(\frac{|\nu_2|}{\nu_1}\right)^{1/2}, 1} \right\}. \tag{45}$$

Using (44) and the energy equation (25) one finds for ε:

$$\varepsilon(t) = \frac{\varepsilon_0}{\left(1 + \frac{\varepsilon_0^{1/2}\, t}{\gamma (\nu_1 |\nu_2|)^{1/4}}\right)^2}, \tag{46}$$

where

$$\varepsilon_0 \equiv \varepsilon(0). \tag{47}$$

The following probability distribution turns out to be stationary for the modified MHD system (see Appendix):

$$\mu \equiv \mu^{\left(\frac{\nu_1}{|\nu_2|}\right)^{1/2}, \left(\frac{|\nu_2|}{\nu_1}\right)^{1/2}, \gamma^2}, \tag{48}$$

where γ is given by (45).

To connect μ with other members of the self-similar family, we choose

$$\xi = \gamma^{-1/2}\, \varepsilon^{1/4}\, (\nu_1 |\nu_2|)^{1/8}, \tag{49}$$

$$\lambda = \gamma^{-1/2}\, \varepsilon^{1/4} (\nu_1 |\nu_2|)^{-3/8}, \tag{50}$$

so that, applying (41) we have

$$\mu^{v_1,|v_2|,\varepsilon} = \sigma_{\gamma^{-1/2}\varepsilon^{1/4}(v_1|v_2|)^{1/8},\,\gamma^{-1/2}\varepsilon^{1/4}(v_1|v_2|)^{-3/8}}\,\mu, \quad (51)$$

or, defining ϑ as

$$\vartheta = \gamma\,\varepsilon^{-1/2}(v_1|v_2|)^{1/4}, \quad (52)$$

we have

$$\mu^{v_1,|v_2|,\varepsilon} = \sigma_{(v_1|v_2|)^{1/4},\,\vartheta^{-1/2},\,(v_1|v_2|)^{-1/4}\vartheta^{-1/2}}\,\mu. \quad (53)$$

5. The Transformed MHD System

The above considerations and (49), (50) and (52) lead to the following change of variables of the MHD equations (7) and (8):

$$\boldsymbol{P} = \left(\frac{(v_1|v_2|)^{1/8}\,\varepsilon^{1/4}}{\gamma^{1/2}}\right)\boldsymbol{M}, \quad (54)$$

$$\boldsymbol{Q} = \left(\frac{(v_1|v_2|)^{1/8}\,\varepsilon^{1/4}}{\gamma^{1/2}}\right)\boldsymbol{N}, \quad (55)$$

$$\boldsymbol{x} = \left(\frac{(v_1|v_2|)^{3/8}\,\gamma^{1/2}}{\varepsilon^{1/4}}\right)\boldsymbol{y}, \quad (56)$$

$$t = \left(\frac{(v_1|v_2|)^{1/4}\,\gamma}{\varepsilon^{1/2}}\right)\tau, \quad (57)$$

where $\boldsymbol{M}$, $\boldsymbol{N}$, $\boldsymbol{y}$ and τ are the new variables.

Using $\varepsilon(t)$ given by (46), the new system of equations is

$$(1-\tau)\frac{\partial \boldsymbol{M}}{\partial\tau} - \frac{1}{2}\{\boldsymbol{M}+(\boldsymbol{y}\cdot\nabla)\,\boldsymbol{M}\} + (\boldsymbol{N}\cdot\nabla)\,\boldsymbol{M}$$
$$= -\nabla r + \left(\frac{v_1}{|v_2|}\right)^{1/2}\nabla^2\boldsymbol{M} \pm \left(\frac{|v_2|}{v_1}\right)^{1/2}\nabla^2\boldsymbol{N}, \quad (58)$$

$$(1-\tau)\frac{\partial \boldsymbol{N}}{\partial\tau} - \frac{1}{2}\{\boldsymbol{N}+(\boldsymbol{y}\cdot\nabla)\,\boldsymbol{N}\} + (\boldsymbol{M}\cdot\nabla)\,\boldsymbol{N}$$
$$= -\nabla r + \left(\frac{v_1}{|v_2|}\right)^{1/2}\nabla^2\boldsymbol{N} \pm \left(\frac{|v_2|}{v_1}\right)^{1/2}\nabla^2\boldsymbol{M}, \quad (59)$$

where

$$r = \frac{\gamma\,q}{\varepsilon^{1/2}\,(v_1|v_2|)^{1/4}}. \quad (60)$$

Or, letting

$$\tau = 1-\varepsilon^{-s}, \quad (61)$$

we have the following system of equations:

$$\frac{\partial \boldsymbol{M}}{\partial s} - \frac{1}{2}\{\boldsymbol{M}+(\boldsymbol{y}\cdot\nabla)\,\boldsymbol{M}\} + (\boldsymbol{N}\cdot\nabla)\,\boldsymbol{M}$$
$$= -\nabla r + \left(\frac{v_1}{|v_2|}\right)^{1/2}\nabla^2\boldsymbol{M} \pm \left(\frac{|v_2|}{v_1}\right)^{1/2}\nabla^2\boldsymbol{N}, \quad (62)$$

$$\frac{\partial \boldsymbol{N}}{\partial s} - \frac{1}{2}\{\boldsymbol{N}+(\boldsymbol{y}\cdot\nabla)\,\boldsymbol{N}\} + (\boldsymbol{M}\cdot\nabla)\,\boldsymbol{N}$$
$$= -\nabla r + \left(\frac{v_1}{|v_2|}\right)^{1/2}\nabla^2\boldsymbol{N} \pm \left(\frac{|v_2|}{v_1}\right)^{1/2}\nabla^2\boldsymbol{M}. \quad (63)$$

Now, that we have the transformed dynamics, we can for completeness relate back their statistics with the statistics of the MHD system. It will be proved in the Appendix that μ given by (48) is a stationary probability of the modified MHD system (62), (63).

6. Correlation Matrix and Energy Spectrum

The correlation matrix is defined as

$$R_{j,k}(\boldsymbol{y}) = \int \frac{1}{|V|}\int_V X_j(\boldsymbol{x}+\boldsymbol{y})\,X_k(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\mu_t, \quad (64)$$

where

$$\boldsymbol{X} = \begin{pmatrix}\boldsymbol{P}\\ \boldsymbol{Q}\end{pmatrix}. \quad (65)$$

It is easily seen that

$$e(\mu_t) = \mathrm{tr}\,\boldsymbol{R}(0) \quad (66)$$

and that

$$\nabla^2\{\mathrm{tr}\,\boldsymbol{R}(\boldsymbol{y})\}|_{\boldsymbol{y}=0} = -\tfrac{1}{2}\{E_1(\mu_t)+E_2(\mu_t)\}. \quad (67)$$

If the scale transformation $\sigma_{\xi,\lambda}$ is applied to $R_{j,k}$, then

$$R_{j,k}\left(\boldsymbol{y};\frac{v_1\xi}{\lambda},\frac{|v_2|\xi}{\lambda},\xi^3\lambda\varepsilon\right)$$
$$= \int\frac{1}{|V|}\int_V X_j(\boldsymbol{x}+\boldsymbol{y})\,X_k(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}(\sigma_{\xi,\lambda}\,\mu_t)$$
$$= \xi^2\int\frac{1}{\lambda^3|V|}\int_{\lambda^3 V} X_j(\boldsymbol{x}+\lambda\boldsymbol{y})\,X_k(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\mu_t$$
$$= \xi^2 R_{j,k}(\lambda\boldsymbol{y};\,v_1,\,|v_2|,\,\varepsilon). \quad (68)$$

Replacing $\boldsymbol{y}$ for $\boldsymbol{y}/\lambda$:

$$R_{j,k}(\boldsymbol{y};\,v_1,\,|v_2|,\,\varepsilon) = \xi^{-2}R_{j,k}\left(\frac{\boldsymbol{y}}{\lambda};\frac{v_1\xi}{\lambda},\frac{|v_2|\xi}{\lambda},\xi^3\lambda\varepsilon\right). \quad (69)$$

Letting for example:

$$\xi^3 \lambda \varepsilon = 1 \tag{70}$$

and

$$\lambda = |y|, \tag{71}$$

then

$$R_{j,k}(y;\, v_1,\, |v_2|,\, \varepsilon) = \left(\frac{\varepsilon}{|y|}\right)^{2/3} R_{j,k}\left(1, \frac{L_1}{|y|}, \frac{L_2}{|y|}, 1\right), \tag{72}$$

where

$$L_1 = \frac{v_1}{\varepsilon^{1/3}}, \tag{73}$$

$$L_2 = \frac{|v_2|}{\varepsilon^{1/3}}. \tag{74}$$

At this point, we have to assume that in the limit of $v, \eta \to 0$, or $v_1, |v_2| \to 0$:

$$\lim_{v,\eta \to 0} R_{j,k}\left(1, \frac{L_1}{|y|}, \frac{L_2}{|y|}, 1\right) = \text{const}, \tag{75}$$

in which case, the correlations (72) agree with those derived in conjunction with the Kolmogorov spectrum (see [3, 4, 5]).

To a certain extent, this spectrum can be derived by doing a Fourier transformation of the trace of the correlation matrix (64) (see [4]) to obtain

$$S(k) = \varepsilon^{2/3}\, k^{-5/3}\, F\left(\frac{k}{k_d}\right), \tag{76}$$

with

$$k_d = \frac{\varepsilon^{1/4}}{(v_1 |v_2|)^{3/8}}. \tag{77}$$

We are now very near to the spectrum of Kolmogorov [2]. However, a last "mild" assumption $F(k/k_d) \approx C$ is necessary at least for the range $k/k_d \ll 1$ in order to justify the exponent of k, the exponent of ε being naturally obtained. Though our function $F(k/k_d)$ is slightly different from the one obtained in [4], we can closely follow [4] for the discussion of our function F, in order to help in the justification of our assumption.

7. Conclusion

This work extends the kind of derivation of the Kolmogorov spectrum done in [4] and [5] based on a Hopf suggestion (see [3], page 120). It uses the mathematical description initiated in [3] without solving the Hopf equation explicitly. In fact, a self-similar family of probability distributions is established in conjunction with the scaling properties of the MHD equations, written in Elsasser variables [7]. This allows to relate the correlation matrix and spectrum of MHD systems having different values of resistivities and viscosities. Already this fact gives the correct exponent 2/3 in the energy rate appearing in the Kolmogorov spectrum $E_k = C\varepsilon^{2/3} k^{-5/3}$. C and the exponent of k can only be obtained after accepting some assumption on an unknown function (see Section 6).

The boundary conditions on the fluid are such that the solutions of the MHD system as well as the turbulence have to decay. The self-similar family of probability distributions allows, however, to find properties of the correlation matrix and the energy spectrum. A complete determination of the spectrum cannot be found without a knowledge of the explicit solutions of the Hopf equation. From the self-similarity condition, however, it is already possible to "almost" derive the Kolmogorov spectrum for 3-d MHD turbulence.

8. Appendix

It is now shown that any homogeneous statistical solution of the MHD equations (7), (8), μ_t, transformed in accordance with (54)–(57), is a homogeneous statistical solution of (62) and (63). In order to show this, let us take

$$\xi = \frac{\gamma^{1/2}}{(v_1 |v_2|)^{1/8} \varepsilon^{1/4}},$$

$$\lambda = \frac{\gamma^{1/2} (v_1 |v_2|)^{3/8}}{\varepsilon^{1/4}},$$

$$s = -\log(1-\tau).$$

Taking into account the dependence of the linear transformation with time, then for any functional Φ:

$$\frac{\mathrm{d}}{\mathrm{d}s} \int \Phi(\boldsymbol{M}, \boldsymbol{N})\, \mathrm{d}\tilde{\mu}_s = \frac{\mathrm{d}}{\mathrm{d}s} \int \Phi(\boldsymbol{M}, \boldsymbol{N})\, \mathrm{d}(\sigma_{\xi,\lambda}\, \mu_t)$$

$$= \frac{\mathrm{d}t}{\mathrm{d}s} \frac{\mathrm{d}}{\mathrm{d}t} \int \Phi(\sigma_{\xi,\lambda} \boldsymbol{M}, \sigma_{\xi,\lambda} \boldsymbol{N})\, \mathrm{d}\mu_t$$

$$= \frac{\mathrm{d}t}{\mathrm{d}s} \left\{ - \int \mathrm{d}\mu_t \left\{ v_1 ((\boldsymbol{M}, \nabla_M \Phi\{\boldsymbol{Z}, \boldsymbol{W}\})) \right.\right.$$

$$\pm |v_2| ((\boldsymbol{N}, \nabla_M \Phi\{\boldsymbol{Z}, \boldsymbol{W}\}))$$

$$+ (\{\boldsymbol{M} \cdot \nabla\}\, \boldsymbol{N}, \nabla_M \Phi\{\boldsymbol{Z}, \boldsymbol{W}\})$$

$$\left.\left. + v_1 ((\boldsymbol{N}, \nabla_N \Phi\{\boldsymbol{Z}, \boldsymbol{W}\})) \pm |v_2| ((\boldsymbol{M}, \nabla_N \Phi\{\boldsymbol{Z}, \boldsymbol{W}\})) \right.\right.$$

$$+(\{\boldsymbol{N}\cdot\nabla\}\,\boldsymbol{M},\nabla_N\Phi\{\boldsymbol{Z},\boldsymbol{W}\})$$
$$-\left(\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{\xi,\lambda}\,\boldsymbol{M},\Phi'_Z\{\boldsymbol{Z},\boldsymbol{W}\}\right)$$
$$-\left(\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{\xi,\lambda}\,\boldsymbol{N},\Phi'_W\{\boldsymbol{Z},\boldsymbol{W}\}\right)\Big|_{\boldsymbol{Z}=\sigma_{\xi,\lambda}\boldsymbol{M},\boldsymbol{W}=\sigma_{\xi,\lambda}\boldsymbol{N}}\Big\}\Big\}.$$

It must be noticed that the scalar transformation used here is the inverse transformation of (49), (50), and thus $\boldsymbol{Z}$, $\boldsymbol{W}$ can be identified with $\boldsymbol{P}$ and $\boldsymbol{Q}$, respectively.

As

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sigma_{\xi,\lambda}\,\boldsymbol{M})=\frac{\varepsilon^{1/2}}{2(v_1|v_2|)^{1/4}\gamma}(\boldsymbol{Z}+\{\boldsymbol{y}\cdot\nabla\}\,\boldsymbol{Z})|_{\boldsymbol{Z}=\sigma_{\xi,\lambda}\boldsymbol{M}}$$

and

$$\frac{\mathrm{d}t}{\mathrm{d}s}=\frac{(v_1|v_2|)^{1/4}\gamma}{\varepsilon^{1/2}},$$

the preceding expression can be written as

$$\frac{\mathrm{d}}{\mathrm{d}s}\int\Phi(\boldsymbol{M},\boldsymbol{N})\;\mathrm{d}\tilde{\mu}_s$$
$$=-\int\mathrm{d}\tilde{\mu}_s\left\{\left(\frac{v_1}{|v_2|}\right)^{1/2}((\boldsymbol{M},\Phi'_M))\right.$$
$$\pm\left(\frac{|v_2|}{v_1}\right)^{1/2}((\boldsymbol{N},\Phi'_M))+(\{\boldsymbol{N}\cdot\nabla\}\,\boldsymbol{M},\Phi'_M)$$
$$-\tfrac{1}{2}(\boldsymbol{M}+\{\boldsymbol{y}\cdot\nabla\}\,\boldsymbol{M},\Phi'_M)+\left(\frac{v_1}{|v_2|}\right)^{1/2}((\boldsymbol{N},\Phi'_N))$$
$$\left.\pm\left(\frac{|v_2|}{v_1}\right)^{1/2}((\boldsymbol{N},\Phi'_N))-\tfrac{1}{2}(\boldsymbol{N}+\{\boldsymbol{y}\cdot\nabla\}\,\boldsymbol{N},\Phi'_N)\right\}.$$

The equation above is the Liouville equation for the system (62), (63), showing that μ_s is a homogeneous statistical solution of the transformed system.

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