IL NUOVO CIMENTO VOL. 107 B, N. 7 Luglio 1992

On the Nonlinear Stability of Dissipative Fluids (*).

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(ricevuto l'1 Marzo 1991; revisionato il 21 Novembre 1991; approvato il 7 Gennaio 1992)

Summary. — A general sufficient condition for nonlinear stability of steady and unsteady flows in hydrodynamics and magnetohydrodynamics is derived. It leads to strong limitations in the Reynolds and magnetic Reynolds numbers. It is applied to the stability of generalized time-dependent planar Couette flows in magnetohydrodynamics. Reynolds and magnetic Reynolds numbers have to be typically less than $2\pi^2$ for stability.

PACS 47.10 – General theory.
PACS 47.20 – Hydrodynamic stability.
PACS 47.65 – Magnetohydrodynamics and electrohydrodynamics.

For incompressible fluids and in particular in hydrodynamics (HD) and magnetohydrodynamics (MHD) the nonlinear terms in the equation of motion are of the quasi-linear type and dissipation is present in the form of material viscosity or resistivity. More precisely if $\underline{u}$ is a many-component vector field in an L^2 function space representing the frame of the fluid motion, $\underline{u}$ will obey an equation of the form

(1) $$\dot{\underline{u}} = A(\underline{u})\,\underline{u} + D\underline{u}\,,$$

where $A(\underline{u})$ is a nonlinear operator depending linearly upon $\underline{u}$ and D is a linear negative definite operator if $\underline{u} = 0$ at the boundary. A simple example is

(2) $$A(\underline{u})\,\underline{u} = \underline{u}\cdot\nabla\underline{u}\,, \qquad D\underline{u} = \nabla^2\underline{u}\,.$$

We assume further that

(3) $$(\underline{u}, A(\underline{u})\,\underline{u}) = 0\,,$$

(*) The authors of this paper have agreed to not receive the proofs for correction.
(**) Partially supported by DAAD and CAPES.

where the scalar product is given by

$$(\underset{\sim}{a}, \underset{\sim}{b}) = \int \underset{\sim}{a} \cdot \underset{\sim}{b} \, \mathrm{d}\tau , \tag{4}$$

the integration being done over the volume occupied by the fluid.

To study the nonlinear stability we split $\underset{\sim}{u}$ in

$$\underset{\sim}{u} = \underset{\sim}{u}_0 + \underset{\sim}{u}_1 , \tag{5}$$

where $\underset{\sim}{u}_1$ is a finite perturbation zero at the boundary and $\underset{\sim}{u}_0$ satisfies

$$\dot{\underset{\sim}{u}}_0 = A(\underset{\sim}{u}_0)\, \underset{\sim}{u}_0 + D\underset{\sim}{u}_0 . \tag{6}$$

The equation for $\underset{\sim}{u}_1$ is then

$$\dot{\underset{\sim}{u}}_1 = A(\underset{\sim}{u}_1)\, \underset{\sim}{u}_1 + L\underset{\sim}{u}_1 , \tag{7}$$

with

$$L\underset{\sim}{u}_1 = A(\underset{\sim}{u}_0)\, \underset{\sim}{u}_1 + A(\underset{\sim}{u}_1)\, \underset{\sim}{u}_0 + D\underset{\sim}{u}_1 . \tag{8}$$

L is a linear operator on $\underset{\sim}{u}_1$ which, in cases like (2), will remain negative definite if $A(\underset{\sim}{u}_0)$ and $\underset{\sim}{u}_0$ are small enough. Taking the scalar product of $\underset{\sim}{u}_1$ with eq. (7) we obtain

$$\frac{1}{2} (\underset{\sim}{u}_1 , \underset{\sim}{u}_1)^{\cdot} = (\underset{\sim}{u}_1 , L\underset{\sim}{u}_1) \tag{9}$$

by virtue of (3). Since all considered quantities are real we have

$$(\underset{\sim}{u}_1 , L\underset{\sim}{u}_1) = (\underset{\sim}{u}_1 , L_\mathrm{s} \underset{\sim}{u}_1), \tag{10}$$

where L_s is the symmetric part of L. Nonlinear stability is then warranted by Lyapunov methods if

$$(\underset{\sim}{u}_1 , L_\mathrm{s} \underset{\sim}{u}_1) < 0 , \tag{11}$$

for all $\underset{\sim}{u}_1$ satisfying $(\underset{\sim}{u}_1 , \underset{\sim}{u}_1)$ finite and $\underset{\sim}{u}_1 = 0$ at the boundary. Expression (11) is a sufficient condition for nonlinear stability. The stability problem is now reduced to the minimization of the Hermitian form $(\underset{\sim}{u}_1 , L_\mathrm{s} \underset{\sim}{u}_1)$. This can always be done ultimately for any flow, numerically using standard Hermitian eigenvalues techniques.

This procedure is known (see [1-4]) for steady HD and MHD flows satisfying

$$A(\underset{\sim}{u}_0)\, \underset{\sim}{u}_0 + D\underset{\sim}{u}_0 = 0 , \tag{12}$$

which is equivalent to (6) for time-dependent flows. Though we did not find it in this general form in the literature, it is likely that it has also been used (see [5]) for unsteady HD flows. We are not aware, however, of the derivation and application of (11) for MHD unsteady flows. The MHD equations are

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{v} \cdot \nabla \boldsymbol{v} + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla p + \nu \nabla^2 \boldsymbol{v}, \tag{13}$$

$$\nabla \cdot \boldsymbol{v} = 0 , \tag{14}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}, \tag{15}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{16}$$

If $\underline{u}$ is defined as

$$\underline{u} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{B} \end{pmatrix}, \tag{17}$$

then $A(\underline{u})$ and D are the matrix operators

$$A(\underline{u}) = \begin{pmatrix} -(\boldsymbol{v} \cdot \nabla) & (\nabla \times \boldsymbol{B}) \times \\ (\boldsymbol{B} \cdot \nabla) & -(\boldsymbol{v} \cdot \nabla) \end{pmatrix}, \tag{18}$$

$$D = \begin{pmatrix} \nu \nabla^2 & 0 \\ 0 & \eta \nabla^2 \end{pmatrix}, \tag{19}$$

with $A(\underline{u})$ verifying condition (3), for normal component of $\underline{u}$ zero at the boundary, and D being negative definite if $\underline{u}$ is zero at the boundary.

Let us illustrate the procedure by studying the nonlinear stability of a time-dependent MHD flow generalizing the time-dependent planar Couette flow. It consists of a fluid bounded by two horizontal plates, the first plate at $z = 0$ and the second at $z = h$, with velocity parallel to the magnetic field and both depending only on one coordinate (z) and the time (t):

$$\boldsymbol{v} = v(z, t)\, \hat{e}_y, \tag{20}$$

$$\boldsymbol{B} = B(z, t)\, \hat{e}_y, \tag{21}$$

satisfying the equations

$$\frac{\partial v}{\partial t} - \nu \frac{\partial^2 v}{\partial z^2} = 0, \tag{22}$$

$$\frac{\partial B}{\partial t} - \eta \frac{\partial^2 B}{\partial z^2} = 0, \tag{23}$$

$$\frac{\partial p}{\partial z} + B \frac{\partial B}{\partial z} = 0. \tag{24}$$

For simplicity special solutions of these equations can be taken as

$$\boldsymbol{v} = \frac{v_0}{\sin \sqrt{\frac{\alpha}{\nu}}\, h} \exp[-\alpha t] \sin \sqrt{\frac{\alpha}{\nu}}\, z \hat{e}_y, \tag{25}$$

$$\boldsymbol{B} = \frac{B_0}{\sin \sqrt{\frac{\alpha}{\eta}}\, h} \exp[-\alpha t] \sin \sqrt{\frac{\alpha}{\eta}}\, z \hat{e}_y, \tag{26}$$

$$p = -\frac{B^2}{2} f(t), \tag{27}$$

with the following boundary conditions:

$$v(0, t) = B(0, t) = 0\,, \tag{28}$$

$$v(h, t) = v_0 \exp[-\alpha t]\,, \tag{29}$$

$$B(h, t) = B_0 \exp[-\alpha t] \tag{30}$$

and $f(t)$ fixed by the boundary conditions on p. In the limit $\alpha \to 0$ this system reduces to a stationary MHD flow. For $B_0 \to 0$ we have the time-dependent Couette flow and when both α and $B_0 \to 0$ we obtain the stationary Couette flow. After calculating L (see eq. (8)) for this case, we obtain its symmetric part L_s

$$L_s = \frac{1}{2}\begin{pmatrix} 2\nu\nabla^2 & 0 & 0 & 0 & -B\dfrac{\partial}{\partial x} & 0 \\ 0 & 2\nu\nabla^2 & 0 & 0 & -B\dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & 2\nu\nabla^2 & 0 & -B\dfrac{\partial}{\partial z} - \dfrac{\partial B}{\partial z} & 0 \\ 0 & 0 & 0 & 2\eta\nabla^2 & 0 & 0 \\ B\dfrac{\partial}{\partial x} & B\dfrac{\partial}{\partial y} & B\dfrac{\partial}{\partial z} & 0 & 2\eta\nabla^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\eta\nabla^2 \end{pmatrix} + \tag{31}$$

$$+\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{\partial v}{\partial z} & 0 & 0 & \dfrac{\partial B}{\partial z} \\ 0 & -\dfrac{\partial v}{\partial z} & 0 & 0 & -\dfrac{\partial B}{\partial z} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{\partial B}{\partial z} & 0 & 0 & \dfrac{\partial v}{\partial z} \\ 0 & \dfrac{\partial B}{\partial z} & 0 & 0 & \dfrac{\partial v}{\partial z} & 0 \end{pmatrix}.$$

In order to examine stability, we calculate I given by

$$I = (\underline{u}_1, L_s \underline{u}_1) = \int \mathrm{d}\tau \Bigg(\nu(v_{1x}\nabla^2 v_{1x} + v_{1y}\nabla^2 v_{1y} + v_{1z}\nabla^2 v_{1z}) + \tag{32}$$
$$+ \eta(B_{1x}\nabla^2 B_{1x} + B_{1y}\nabla^2 B_{1y} + B_{1z}\nabla^2 B_{1z}) +$$
$$+ \left(\frac{\partial v}{\partial z}(-v_{1y}v_{1z} + B_{1y}B_{1z} + \frac{\partial B}{\partial z}(v_{1y}B_{1z} - v_{1z}B_{1y}\right)\Bigg).$$

We suppose for convenience that v_0, $B_0 > 0$, $0 < h < \sqrt{\nu/\alpha}\,\pi/2$ and $0 < h < \sqrt{\eta/\alpha}\,\pi/2$, which guarantees that $(\partial v/\partial z) > 0$ and $(\partial B/\partial z) > 0$. To satisfy condition (11) for all $\underline{u}_1$

we make a first estimate of I using

$$-\left(\frac{\partial v}{\partial z}\right) v_{1y} v_{1z} < \left(\frac{\partial v}{\partial z}\right) \frac{1}{2} (v_{1y}^2 + v_{1z}^2) < \left(\frac{\partial v}{\partial z}\right)_{\mathrm{m}} \frac{1}{2} (v_{1y}^2 + v_{1z}^2), \tag{33}$$

where $(\partial v/\partial z)_{\mathrm{m}}$ is the maximum of $(\partial v/\partial z)$ with respect to z and t, which occurs at $t = 0$, $z = 0$. Similar estimates can be done for the other cross terms so that

$$I < I_{\mathrm{m}} = \int \mathrm{d}\tau (\nu (v_{1x} \nabla^2 v_{1x} + v_{1y} \nabla^2 v_{1y} + v_{1z} \nabla^2 v_{1z}) + \eta(B_{1x} \nabla^2 B_{1x} + B_{1y} \nabla^2 B_{1y} + B_{1z} \nabla^2 B_{1z}) + C_{\mathrm{m}} (v_{1y}^2 + v_{1z}^2 + B_{1y}^2 + B_{1z}^2)), \tag{34}$$

with

$$c_{\mathrm{m}} = \left(\frac{\partial v}{\partial z}\right)_{\mathrm{m}} + \left(\frac{\partial B}{\partial z}\right)_{\mathrm{m}}, \tag{35}$$

$$\left(\frac{\partial v}{\partial z}\right)_{\mathrm{m}} = \frac{v_0}{\sin \sqrt{\frac{\alpha}{\nu}}\, h} \sqrt{\frac{\alpha}{\nu}}, \tag{36}$$

$$\left(\frac{\partial B}{\partial z}\right)_{\mathrm{m}} = \frac{B_0}{\sin \sqrt{\frac{\alpha}{\eta}}\, h} \sqrt{\frac{\alpha}{\eta}}. \tag{37}$$

Now we look for the extremum of I_{m} subject to the condition $(\underline{u}_1, \underline{u}_1) =$ finite.

$$\delta I' = \delta(I_{\mathrm{m}} - \lambda(\underline{u}_1, \underline{u}_1)) = 0, \tag{38}$$

where λ is the Lagrange multiplier. This leads to the following system of equations:

$$\nabla^2 v_{1x} - \frac{\lambda}{\nu} v_{1x} = 0, \tag{39}$$

$$\nabla^2 v_{1y} + \left(\frac{1}{2\nu} C_{\mathrm{m}} - \frac{\lambda}{\nu}\right) v_{1y} = 0, \tag{40}$$

$$\nabla^2 v_{1z} + \left(\frac{1}{2\nu} C_{\mathrm{m}} - \frac{\lambda}{\nu}\right) v_{1z} = 0, \tag{41}$$

$$\nabla^2 B_{1x} - \frac{\lambda}{\eta} B_{1x} = 0, \tag{42}$$

$$\nabla^2 B_{1y} + \left(\frac{1}{2\eta} C_{\mathrm{m}} - \frac{\lambda}{\eta}\right) B_{1y} = 0, \tag{43}$$

$$\nabla^2 B_{1z} + \left(\frac{1}{2\eta} C_{\mathrm{m}} - \frac{\lambda}{\eta}\right) B_{1z} = 0. \tag{44}$$

Fourier analysing in x and y

$$\underline{u}_1 = \sum_k A_k(z) \exp[i\boldsymbol{k}\cdot\boldsymbol{r}], \tag{45}$$

where

$$\boldsymbol{k} = k_x \hat{e}_x + k_y \hat{e}_y, \tag{46}$$

$$\boldsymbol{r} = x\hat{e}_x + y\hat{e}_y \tag{47}$$

and calling each component of the vector $A_k(z)$ as

$$(A_k(z))_j = A_j \tag{48}$$

the system of eqs. (38)-(43) can be reduced to

$$\frac{d^2 A_1}{dz^2} - \left(\frac{\lambda}{\nu} + k^2\right) A_1 = 0, \tag{49}$$

$$\frac{d^2 A_2}{dz^2} + \left(\frac{1}{2\nu} C_m - \frac{\lambda}{\nu} - k^2\right) A_2 = 0, \tag{50}$$

$$\frac{d^2 A_3}{dz^2} + \left(\frac{1}{2\nu} C_m - \frac{\lambda}{\nu} - k^2\right) A_3 = 0, \tag{51}$$

$$\frac{d^2 A_4}{dz^2} - \left(\frac{\lambda}{\eta} + k^2\right) A_4 = 0, \tag{52}$$

$$\frac{d^2 A_5}{dz^2} + \left(\frac{1}{2\eta} C_m - \frac{\lambda}{\eta} - k^2\right) A_5 = 0, \tag{53}$$

$$\frac{d^2 A_6}{dz^2} + \left(\frac{1}{2\eta} C_m - \frac{\lambda}{\eta} - k^2\right) A_6 = 0. \tag{54}$$

Since the boundary conditions for the perturbations are

$$\boldsymbol{v}_1(x, y, 0, t) = \boldsymbol{B}_1(x, y, 0, t) = 0, \tag{55}$$

$$\boldsymbol{v}_1(x, y, h, t) = \boldsymbol{B}_1(x, y, h, t) = 0, \tag{56}$$

the nontrivial solutions of this system which satisfy the boundary conditions (54) are sine solutions. To satisfy also the other boundary conditions (55) we obtain some restrictions on λ. When the maximum value of λ is negative, the system is stable, this can occur in two ways.

For $\nu < \eta$ the system is stable if

$$\mathrm{Re}\, \frac{\sqrt{\frac{\alpha}{\nu}}\, h}{\sin\sqrt{\frac{\alpha}{\nu}}\, h} + S\, \frac{\sqrt{\frac{\alpha}{\eta}}\, h}{\sin\sqrt{\frac{\alpha}{\eta}}\, h} < 2\pi^2, \tag{57}$$

where

$$\mathrm{Re} = \frac{v_0 h}{\nu}\,, \tag{58}$$

$$S = \frac{B_0 h}{\nu}\,. \tag{59}$$

For $\eta < \nu$ the system is stable if

$$\mathrm{Re}_\mathrm{m} \frac{\sqrt{\frac{\alpha}{\nu}}\, h}{\sin \sqrt{\frac{\alpha}{\nu}}\, h} + S_\mathrm{m} \frac{\sqrt{\frac{\alpha}{\eta}}\, h}{\sin \sqrt{\frac{\alpha}{\eta}}\, h} < 2\pi^2\,, \tag{60}$$

where

$$\mathrm{Re}_\mathrm{m} = \frac{v_0 h}{\eta}\,, \tag{61}$$

$$S_\mathrm{m} = \frac{B_0 h}{\eta}\,. \tag{62}$$

In the limit $\alpha \to 0$ (steady MHD flow) we obtain

$$\mathrm{Re} + S < 2\pi^2\,, \qquad \text{for } \nu < \eta\,, \tag{63}$$

and

$$\mathrm{Re}_\mathrm{m} + S_\mathrm{m} < 2\pi^2\,, \qquad \text{for } \eta < \nu\,. \tag{64}$$

For the time-dependent Couette flow ($B_0 \to 0$), we have

$$\mathrm{Re} \frac{\sqrt{\frac{\alpha}{\nu}}\, h}{\sin \sqrt{\frac{\alpha}{\nu}}\, h} < 2\pi^2 \tag{65}$$

and for the stationary Couette flow (α, $B_0 \to 0$)

$$\mathrm{Re} < 2\pi^2\,. \tag{66}$$

It should be mentioned that for the stationary Couette flow the last condition is also obtained without introducing the estimate $I < I_\mathrm{m}$.

The critical value $2\pi^2 \simeq 19.7$ for the Reynolds number calls for some comments as indirectly suggested by one of the referees. The extremalization of I for the Couette flow in HD has been done in the literature by constraining the variations to be divergence-free (see, *e.g.* [6]). As a consequence of that, the critical value of 20.7 is found. This gain of 5% in the critical value is paid by a very sophisticated derivation which would not be tractable in the case of the generalized unsteady MHD Couette flow considered here. This justifies our procedure.

The sufficient condition (11) is general and robust, but also too stringent. It is fulfilled in HD and MHD only if the Reynolds and magnetic Reynolds numbers are small enough. Since viscosity and resistivity especially for hot plasmas are small, condition (11) would allow only a very low level of electrical currents and flows.

Linear stability analysis and experimental evidence, however, seem to show that, in some cases, values for currents and flows far beyond those allowed by condition (11) occur without any sign of gross instabilities. It will be, however, much more difficult to do the nonlinear stability theory for such situations and, in contrast with the present method, it is likely that it may have to be done differently for each case.

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