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Forced baroclinic ocean motions: II. The linear equatorial bounded case

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ABSTRACT

This paper extends the results of Cane and Sarachik (1976) to an ocean bounded by two meridians. A complete solution is obtained for the asymptotic linear inviscid response to wind stress and thermal forcings independent of longitude, switched on at $t=0$ and steady thereafter. The mathematics is greatly simplified by building on the results of the earlier paper. The form of the solution is relatively simple: in addition to the unbounded response and inertia-gravity waves it consists of quasi-geostrophic Rossby modes that form the eastern boundary response, a western boundary layer, and equatorial Kelvin waves. Some points concerning the eastern boundary reflection of a Kelvin wave are clarified.

It is shown that the nature of the spin-up process is very different according to whether or not Kelvin waves are generated as part of the response. The Kelvin wave will be present if the forcings include a zonal wind stress component that is symmetric about the equator or a meridional wind stress that is anti-symmetric. When the Kelvin waves are absent the spin-up is effected entirely by the Rossby waves emanating from the eastern boundary. As more and more of these reach an interior point, the circulation gets closer and closer to a steady state. This spin-up proceeds from east to west, and, because the more rapid Rossby waves are more equatorially confined, the steady state is approached more slowly the further the point is from the equator.

For those symmetries that allow a Kelvin wave the Rossby waves again act to bring the currents and sea surface tilt toward a steady state. In addition, the Kelvin waves have a net (meridionally integrated) eastward mass flux and the Rossby waves a net westward mass flux. As a result there is a persistent and substantial oscillation of mass zonally across the basin delaying the adjustment to a steady state even at the equator. The characterization of spin-up given in the preceding paragraph applies regardless of whether the wind stress is global, equatorial or extra-tropical and whether or not its curl is nonzero.

1. Introduction

This paper presents results on the spin-up of an equatorial ocean governed by linear inviscid dynamics. Because the response time of the baroclinic modes in an

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equatorial ocean is on the order of a month while the corresponding mid-latitude baroclinic time is on the order of years, we may expect to see baroclinic responses to suddenly applied forces only at low latitudes. Since equatorial currents are swift and narrow, viscous and nonlinear advective effects will usually be significant. Nevertheless, a model based on a linearization about a resting basic state is still useful for the insight it supplies into the more realistic viscous, nonlinear dynamics. In some cases linear analysis may be applied to real oceanic situations: Lighthill (1969) and Anderson and Rowlands (1976b) have done so for the Somali Current as have Godfrey (1975), Hurlburt *et al.* (1976), and McCreary (1976) for El Niño and O'Brien and Hurlburt (1975) for the equatorial jet in the Indian Ocean.

The results we will present *are* directly relevant to numerical modelling because the same set of simplifications are shared by the analysis and the numerical models —the real ocean is not so cooperative. Most numerical models are spun up from rest; yet in the past most theoretical analyses have been focused on the final steady states whose dynamics are viscous and nonlinear. It is our belief that much insight and understanding of the dynamics of simple numerical models (and, one hopes, real oceans) can be gained by linear analysis of the initial stages of spin-up and careful comparison with the spin-up of viscous nonlinear numerical models. In this way the effects of viscosity on the linear response, and the gradual introduction of the nonlinearities may be more fully understood. Such a program has been carried out with a numerical model described by one of us (Cane, 1975, 1976).

The problem to be treated in this paper is the linear inviscid baroclinic response of an equatorial ocean, bounded by meridional boundaries $x=X_E$ at the east and $x=0$ at the west, to switched-on wind and mass forcings constant with longitude over the basin. Zonal boundaries at the north and south tend to be extraequatorial (with the important exception of the Gulf of Guinea—see Hickie, 1976) and therefore play a secondary role in equatorial spin-up. This is discussed at length in Sec. 4 of Cane and Sarachik (1976), henceforth referred to as I. The mathematical problem to be solved is the linear shallow water equations on an equatorial beta plane. Scaling distance and time equatorially (see I, also Matsuno, 1966; Blandford, 1966) gives them the form

$$u_t - yv + h_x = F(y)H(t) \tag{1a}$$

$$v_t + yu + h_y = G(y)H(t) \tag{1b}$$

$$h_t + u_x + v_y = Q(y)H(t). \tag{1c}$$

where H denotes the Heaviside step function. The boundary conditions are

$$u = v = h = 0 \text{ at } y = \pm\ \infty \tag{1d}$$

$$u = 0 \text{ at } x = 0 \text{ and } x = X_E \tag{1e}$$

and the initial conditions are $u = v = h = 0$ at $t = 0$. The notation is canonical and has been described in I.

Our overall approach to this problem follows Lighthill (1969). First the unbounded, inhomogeneous problem (1a)-(1d) is solved. Next, the boundary conditions (1e) are satisfied by adding appropriate free modes (i.e. solutions to the homogeneous problem (1a)-(1d) with $F=G=Q=0$). The first task was accomplished in I. By making use of other results of I the second step may be done using only elementary mathematics. No transforms need be inverted; it is only necessary to do some algebra to determine the proper amplitude of each free mode.

The free modes that will be used are large t asymptotic forms so the solutions we will obtain are approximate. One of the major purposes of I was to clarify the nature of these approximations; it is not necessary to repeat the analysis here. Suffice it to say that they capture the physically interesting part of the total solution; the part that is not considered here consists primarily of small scale wiggles at wave fronts. Moreover, the forms we employ here are analytically tractable and relatively easy to comprehend. More complete solutions would be unwieldy to calculate with and difficult to gain insight from. [See Anderson and Rowlands (1976a) for an example of a complete account of a single reflection.]

We have tried to make this paper independent of I but certain *caveats* apply. As stated in the last paragraph, the asymptotic results of I will be used here without further justification; the reader concerned about their validity or derivation should consult I. An important virtue of finding the response to step functions in time is that the response to more general structures may be found in a number of ways: other time structures may be found by convolution; the asymptotic response to t^{s+1} may be found from that to t^s (starting with $s=0$) by integrating with respect to t; once this is done a more general $f(t)$ may be fit by polynomials or straight line segments. However, when the step function response is approximate, one must understand the nature of the approximations employed in order to use correctly any of these strategies. Thus, while this paper may be read without knowledge of I, the earlier paper enriches the present results. In a similar vein, we note that in I we found the response to a forcing of the form $F(y)H(x-x_*)H(t)$, i.e. a step in x as well as t. This allows general spatial structures to be calculated. In the present paper we consider in detail only forcings that are constant with longitude because the response is sufficiently representative of boundary effects. The $H(x-x_*)$ response may be calculated in the same manner beginning from the unbounded response presented in I, Sec. 6.

We further found in I that the physics and mathematics of the planetary waves could be illustrated more lucidly in the context of the barotropic vorticity equation whose dispersion relation is similar to that for the planetary wave modes of the full shallow water equations. The essential dispersive features of spin-up are present in the barotropic case so a brief discussion of this case serves to preface the baroclinic results.

The dominant barotropic asymptotic response in an unbounded ocean to a

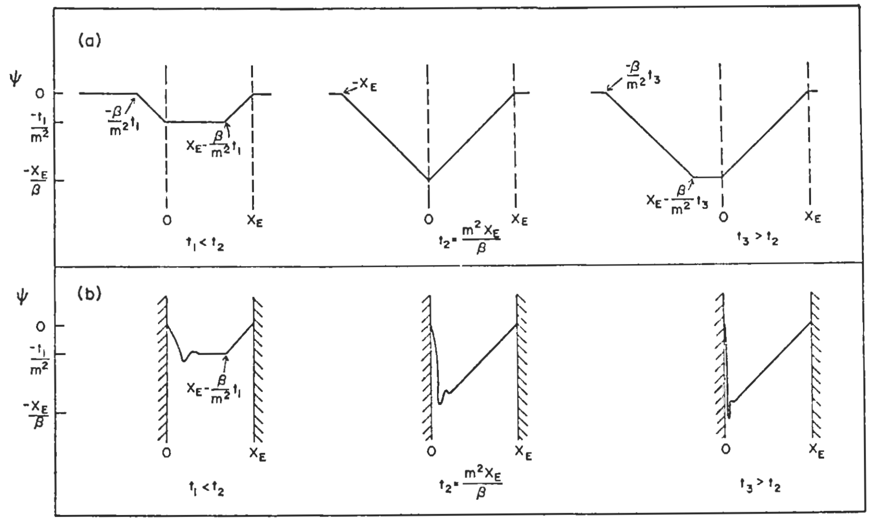


Figure 1. Sketch of the dominant barotropic asymptotic response to a switched on wind stress curl of top hat form: curl $\tau = 1$ for $0<x<X_E$; curl $\tau = 0$ otherwise. (a) Response in an unbounded ocean. (b) Response in an ocean bounded by meridians at $x=0$, $x=X_E$.

switched on wind stress curl of top hat form $T(0; x; X_E) = H(x-0) - H(x-X_E)$ is found by the methods of I to be

$$\psi(x,t) = \beta^{-1}[(x-X_E)T(0;x;X_E) - (x-X_E+\beta m^{-2}t)T(0;x+\beta m^{-2}t;X_E) - X_E T(-\beta m^{-2}t;x;0)]$$

which consists of a secularly growing streamfunction to the west of a front starting at X_E at $t = 0$ and propagating westward with velocity $-\beta m^{-2}$. There is steady Sverdrup flow $\psi = \beta^{-1}(x-X_E)$ to the east of this front. Spin-up is essentially complete when the front reaches $x=0$ at a time $\beta^{-1}m^2X_E$, though residual transients continue to die away as $t^{-1/2}$. If the ocean is unbounded, the return flow occurs *outside* the forcing region, to the west of $x=0$. While boundary layers are present at the edges of the forcing region, they are asymptotically unimportant (of $0(t^{-3/4})$) and carry no net meridional mass flux—they therefore play no role in the meridional mass conservation. The evolution of the flow is depicted schematically in Figure 1a.

If the ocean is bounded (Fig. 1b), the hyperbolic response is given by only the part of ψ between $x=0$ and $x=X_E$, and the mass clearly has to be returned *inside* the basin; consequently the streamfunction must be brought to zero at $x=0$. This is accomplished by a boundary layer contribution $\psi^B=-X_E\beta^{-1}J_0\ (2\sqrt{\beta xt})$. ψ^B carries a net meridional mass flux $-X_E/\beta$ to balance the entire interior flux $+X_E/\beta$. Again spin-up is essentially complete at time $\beta^{-1}m^2X_E$ but the spun-up state here has a continuously thinning boundary layer of width $0(1/t)$ along the western boundary. (See Anderson and Gill, 1975, for a numerical simulation.)

The barotropic vorticity equation sees only the gradient of the Coriolis parameter, which is constant on a beta plane. All mass rearrangement takes place in a meridional distance determined only by the meridional scale of the forcing.

We will find that the equatorial baroclinic spin-up is different in two primary respects. The first is the very large influence of the Kelvin wave, and the second is the meridional behavior of the spin-up. We will show that the Kelvin wave plays a commanding role in the mass redistribution necessary for certain types of spin-up. Further, we will find that because the speed of nondispersive Rossby waves decreases with increasing meridional mode number, the equator tends to spin up first, with higher latitudes taking longer to spin up. A more general overall difference is that the baroclinic shallow water equations spin up to wind stress with or without curl.

The plan of the remainder of the paper is as follows. Sec. 2 reviews the unbounded solution obtained in I and establishes notation. The calculation of the boundary response to the individual components of this solution is carried out in Sec. 3. The procedure is an extension of Moore's method of calculating reflections of an incident wave at a single frequency. Sec. 4 contains a general discussion of equatorial spin-up outlining the contribution of various types of motions. Sec. 5 describes calculations of spin-up due to general types of wind stress forcings in order of increasing complexity and shows the similarities and differences among them. Particular attention is given to the redistribution of mass and the approach to a final state. Sec. 6 completes and summarizes our results.

2. Review of previous results

The free wave solutions to (1) are of four types: Kelvin waves, Rossby waves, inertia-gravity waves and mixed Rossby-gravity waves. By "free waves" we mean solutions to (1) with $F=G=Q=0$ that have $u, v, h \propto e^{i(kx-\omega t)}$. An extended account of the characteristics of these waves may be found in Moore and Philander, 1976. Here we will only recall for the reader those properties important in what follows.

The Rossby and inertia-gravity waves satisfy

$$k = k_n^{\pm}(\omega) \equiv -\frac{1}{2\omega} \pm \left[\omega^2 + \frac{1}{4\omega^2} - (2n+1)\right]^{\frac{1}{2}} . \tag{2a}$$

where n is a positive integer. The mixed mode will be labelled by $n=0$; its dispersion relation is

$$k_0 = \omega - \omega^{-1} . \tag{2b}$$

The Kelvin wave, labelled $n=-1$, is nondispersive:

$$k_{-1} = \omega . \tag{2c}$$

The label n indexes the meridional structure of the waves (and the modes to be introduced below). Those with n even have u and h components which are antisymmetric and v components which are symmetric about the equator; those indexed by odd n have opposite symmetries. Also, the smaller n is, the more equatorially confined is the mode. The short wavelength ($k<<0$) Rossby waves are approximately nondivergent with their v and h components in geostrophic balance while the long wave Rossby waves have v small and u and h in geostrophic balance (as does the Kelvin wave).

The phase velocity ω/k and the group velocity $C_g=\partial\omega/\partial k$ are both eastward at all frequencies for the Kelvin wave. For the $n=0$ wave (often called the mixed mode) the phase velocity may be eastward or westward but the group velocity is always eastward. For the Rossby waves the phase velocity is always westward while for the inertia-gravity waves it may be eastward or westward. For

$$(n+1)^{1/2} - n^{1/2} < \omega < (n+1)^{1/2} + n^{1/2}$$

the $k_n^{\pm}$ are complex in which case the modes are trapped. If the $k_n^{\pm}$ are real, then one mode has a group velocity to the west and the other group velocity to the east. (If $k_n^{+} = k_n^{-}$ then the group velocity is zero.) For the Rossby waves (ω small), the group velocity is eastward for the short waves ($k = k_n^{-} <<0$) and westward for the long waves. Finally, we note that for a given zonal wave number the larger the value of n the smaller the group velocity.

In I [Eq. 51] we found that the response of an unbounded ocean to a vector of forcing functions $\mathbf{F}=(F,G,Q)$ independent of x switched on at $t=0$ and steady thereafter [that is, the solution to (1a)-(1d)] may be written

$$\mathbf{u} \equiv (u,v,h) = (u^{(1)}(y)t, v^{(1)}(y), h^{(1)}(y)t) + (u^{(2)}(y), 0, h^{(2)}(y)) + \mathbf{u}_I(y,t) \tag{3}$$

where

$$(u^{(1)},0,h^{(1)}) = d_{-1}\mathbf{M}_{-1}(y) + \sum_{n=1}^{\infty} r_n\mathbf{R}_n(y) \ , \tag{4}$$

$$v^{(1)} = - \sum_{n=0}^{\infty} (2n+1)^{-1} d_n\psi_n(y) \ , \tag{5}$$

$$(u^{(2)},0,h^{(2)}) = \sum_{n=0}^{\infty} (2n+1)^{-1} g_n\mathbf{W}_n(y), \tag{6}$$

and, with $m=(2n+1)^{1/2}$

$$\mathbf{u}_I = \sum_{n=0}^{\infty} m^{-2}\left[\mathbf{W}_n(y)+\mathbf{V}_n(y)\frac{\partial}{\partial t}\right][m^{-1}d_n \sin mt + g_n \cos mt] \ . \tag{7}$$

The vectors appearing in (4)-(7) give the meridional structure of each mode; for $n \geq 0$

$$\mathbf{W}_n(y) = (y\psi_n, 0, -d\psi_n/dy) \ ,$$
$$\mathbf{V}_n(y) = (0, \psi_n, 0) \ ,$$
$$\mathbf{M}_n(y) = (-d\psi_n/dy, 0, y\psi_n) \ , \tag{8}$$
$$\mathbf{R}_n(y) = [4n(n+1)]^{-1} \, [(2n+1)\mathbf{M}_n - \mathbf{W}_n] \ ;$$

and $\mathbf{M}_{-1}(y) = 2^{-1/2}(\psi_0, 0, \psi_0)$
where the $\psi_n(y)$ are the (normalized) Hermite functions [see I, Eq. (4a) or Moore and Philander, 1976]. The amplitudes of the terms in (4)-(7) depend on the forcing functions; with the notation $(A)_n \equiv \int_{-\infty}^{+\infty} A\psi_n dy$,

$$d_{-1} = 2^{-1/2}(F+Q)_0 \tag{9a}$$

$$d_n = (yF + dQ/dy)_n \tag{9b}$$

$$e_n = (dF/dy + yQ)_n \tag{9c}$$

$$g_n = (G)_n \tag{9d}$$

$$r_n = e_n - (2n+1)^{-1} d_n \ . \tag{9e}$$

The characteristics of this response were discussed in I, Sec. 5 and will only be touched upon briefly here. The oscillatory terms given by (7) are free inertia-gravity waves needed to satisfy the initial conditions. The terms superscripted with a "1" are nonzero only if the zonal wind stress F or the heating Q is nonzero; the secular growth in (3) is the result of resonantly exciting the $k=0$ Rossby and Kelvin waves. Note that $\mathbf{M}_{-1}$ and the $\mathbf{R}_n$ all have u and h components in geostrophic balance. Terms superscripted with a "2" are nonzero only if the meridional wind stress G is nonzero. As a general rule, with $Q=0$, the time-growing part of the response tends to be more equatorially confined than is the (smooth) zonal wind system that forces it. At the equator a zonal wind causes an accelerating current $u^{(1)}t = F(0)t$, while a meridional wind is balanced by the sea surface setup: $h_y^{(2)} = G(0)$. As mid-latitudes are approached ($y \rightarrow \pm \infty$) the currents take on the wind drift values: $u^{(2)} \sim G/y$, $v^{(1)} \sim -F/y$.

3. Calculation of the effect of meridional boundaries

We now wish to describe how the ocean's response is modified by the presence of meridional walls at $x=0$ and $x=X_E$. We begin at the point where the forced motions in the absence of boundaries have been calculated. The task that remains is that of calculating the boundary response to these motions. That is, we seek the free solutions of the shallow water equations (1) that are required to reduce the normal velocity to zero at the walls.

We may think of boundaries as modifying the unbounded forced response in two distinct ways. The first is as a barrier to incident motions: any part of the oceanic response bringing energy into a boundary must give rise to a reflection to carry the incident energy away from the boundary. The second is as a cutoff of the forcing: for example, a western boundary at $x=0$ has the effect of modifying the forcing by multiplying it by a step function $H(x)$ thus switching it off for $x<0$. The unbounded solution for time t at a point $x>0$ will have to be modified insofar as it depends on motions that originated at points $x<0$ which are now outside the basin. The two influences of boundaries will appear often in the following sections.

Moore (see Moore and Philander, 1976) has given a method for calculating the reflection of an incoming wave. Some of the qualitative features of these reflections may be noted. Each incident wave excites a series of waves. A mode incident on a western boundary excites a response which is as equatorially confined as it, itself, is. Unlike the mid-latitude situation, a mixed mode or Kelvin wave will be part of the response. The latter propagates away from the boundary quickly; the former remains near the western side, though it shows some effects extending into the basin. A mode incident on an eastern boundary excites a response which is less equatorially confined than itself. Extraequatorially, this response asymptotes to a coastal Kelvin wave (Moore, 1968). The more equatorially confined parts of the response propagate away from the boundary the most rapidly.

Moore's method generalizes without modification to allow the calculation of the boundary response to any zonal velocity as long as it is oscillating at a single frequency. It may be extended to a motion with arbitrary time structure by analyzing this structure into its frequency spectrum, calculating the boundary response as a function of frequency, and then synthesizing over all frequencies to obtain the time dependent boundary response. In essence, one begins by taking the Laplace transform of the initial motion and finally obtains the time dependent response by inverting the resulting Laplace transformed form of the response. (See Lighthill, 1969; Anderson and Rowlands, 1976a,b).

However, for the time dependences that we need to consider the results of I may be used to bypass these steps. For a forcing that is independent of x (or a step function of x) the only (asymptotically) important motions reaching the boundaries have zonal velocities there that take one of the forms

$$u(y,t) = u(y)t^s, \qquad s=0 \text{ or } s=1 \; ; \tag{10}$$

$$u(y,t) = u(y) \sin mt \text{ or } u(y) \cos mt \tag{11}$$

The time dependences in (11) are just the sum of waves at the frequencies $\pm m$ so Moore's method applies directly. For the steady or secularly growing motions given by (10) we consider the western and eastern boundaries separately.

a. Western boundary response. If the incoming zonal velocity has the form (10) then the response $\mathbf{u}^W$ must be a sum of free solutions of (1) carrying energy east-

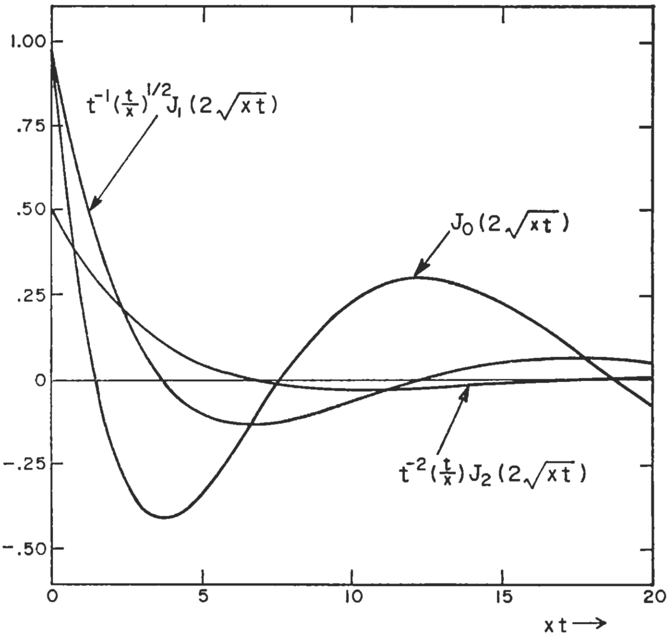


Figure 2. Functions giving the x,t dependence in the western boundary layer.

ward and varying like t^s at the western boundary x=0. By making use of I, Sec. 6 (Eq. 58) we immediately conclude that the response consists of a Kelvin mode plus a sum of terms each of which is a synthesis of short wavelength Rossby waves (including the mixed mode). Specifically,

$$\mathbf{u}^W(x,y,t) = b_{-1}H(t-x)\,(t-x)^s\,\mathbf{M}_{-1}(y) + \mathbf{u}^B(x,y,t) \qquad (12a)$$

with

$$\mathbf{u}^B(x,y,t) \approx \sum_{n=0}^{\infty} b_n H(c_n t-x)\,\{(t/x)^{s/2} J_s\,(2\sqrt{xt})\mathbf{M}_n(y) - (t/x)^{(s+1)/2} J_{s+1}\,(2\sqrt{xt})\mathbf{V}_n(y)\}\ . \qquad (12b)$$

Here J_s is the Bessel function of the first kind of order s. While the Kelvin mode term is exact, the Rossby modes are the leading terms in the asymptotic expansion in t and apply only for $x \lesssim c_n t$, where $c_n \approx [8(2n+1)]^{-1}$ is the maximum eastward group velocity of the nth mode. For $x \gtrsim c_n t$ the amplitude of the solution is asymptotically small and takes a different form; see I. The Rossby modes are trapped near the western boundary in the sense that most of their amplitude is in an ever-thinning region $0 \leqslant x \lesssim (4t)^{-1} j_{s+1}^2$ where j_{s+1} is the first zero of J_{s+1} (Fig. 2). Note that there is some recirculation in the western boundary current (i.e. if v is northward along the coast there will be southward flow further offshore). This follows the form of v (Fig. 2) and is required to give conservation of potential vorticity in the boundary current.

To highest order (in t/x) $\mathbf{u}^B$ is nondivergent with its meridional velocity component in geostrophic balance:

$$\mathbf{u}^B \equiv (u^B,v^B,h^B) = \left[-\frac{\partial}{\partial y} \ , \ \frac{\partial}{\partial x} \ , y \right] \left\{ (t/x)^{s/2} J_s\,(2\sqrt{xt}) \sum_{n=0}^{\infty} b_n \psi_n(y) \right\} \tag{13}$$

Therefore its components satisfy

$$-yv^B + h_x{}^B = 0 \ , \tag{14a}$$

$$v_t{}^B + yu^B + h_y{}^B = 0 \ , \tag{14b}$$

$$u_x{}^B + v_y{}^B = 0 \ , \tag{14c}$$

$$v_{xt}{}^B + v^B = 0 \ . \tag{14d}$$

The coefficients b_n may be calculated readily by applying Moore's algorithm in the limit $\omega \to 0$, $k \approx \omega^{-1}$; for example, if

$$u(y,t) = 2^{-1/2}(J+1)^{1/2}\psi_{J+1}(y)t^s \tag{15}$$

is the incident normal velocity at $x=0$ then

$$b_n = -\alpha_n{}^J \tag{16a}$$

where

$$\alpha_n{}^J \equiv \begin{cases} 0 & \text{if } J \neq n \bmod 2 \text{ or } n > J \\ 1 & \text{if } J = n \\ \left[\dfrac{J}{J-1} \cdot \dfrac{J-2}{J-3} \cdot \ \ldots \ \cdot \dfrac{n+2}{n+1} \right]^{1/2} & J > n \geq 0 \end{cases} \tag{16b}$$

and $\alpha_{-1}{}^J \equiv \alpha_1{}^J$.

The α's will often enter in taking projections since

$$(y)_n \equiv \int_{-\infty}^{+\infty} y\psi_n(y)dy = 2\pi^{1/4}\alpha_1{}^n \ . \tag{17}$$

We can also determine $\mathbf{u}^B$ without performing a modal decomposition in y by making use of (14). If the incident velocity at $x=0$ has the form (10) then the boundary condition at $x=0$ requires that

$$-u(y)t^s = u^B(0,y,t) + 2^{-1/2}b_{-1}t^s\psi_0(y) \ . \tag{18}$$

It follows from the continuity equation (14c) and the boundary conditions $v^B=0$ at $y = \pm\infty$ and $u^B=0$ at $x = +\infty$ that, for all x and t, $\int_{-\infty}^{+\infty} u^B(x,y,t)dy = 0$. Hence, integrating (18)

$$-\int_{-\infty}^{+\infty} u(y)dy = \int_{-\infty}^{+\infty} u^B(0,y,t)dy + 2^{-1/2}b_{-1} \int_{-\infty}^{+\infty} \psi_0(y)dy = \pi^{1/4}b_{-1} \tag{19}$$

Once b_{-1} has been determined from (19) $u^B(0,y,t)$ is given by (18).

Eq. (19) also reveals an important property of the western boundary response: to leading order all of the incoming zonal mass flux is reflected in the Kelvin mode. This fact is crucial to the analysis of the spin-up process presented in Sec. 5. The boundary trapped modes provide no net zonal mass flux though these modes do transport the incoming mass meridionally. This makes it possible for the Kelvin mode to return the net incoming mass eastward regardless of the meridional extent of the incident zonal flow. Of course, it is possible to have a large zonal flow at some latitudes without having any net mass flux; for example, a westward flow south of the equator and an equal eastward flow north of it. In such a case the boundary motions provide the meridional transport needed to close the fluid circuits. This transport may be found from the interior solution [Eqs. (4), (6)] without explicitly calculating the boundary layer structure. For example, the net transport V of the Somali current due to a (symmetric) southerly wind (i.e. the Somali jet) is

$$V(y) \equiv \int_0^\infty v^B(x',y)dx' = \int_{-\infty}^{y} u^B(o,y')dy' = -\int_{-\infty}^{y} u^{(2)}(y')dy' ,$$

where $u^{(2)}$ is the function of the meridional wind stress defined by (6).

For Eqs. (14)*ff* to apply u need not have the special form (10). What is required is that the boundary response exclusive of the Kelvin mode be composed of modes from the lower left-hand corner of the dispersion curve (I, Fig. 1) where $\partial/\partial x >> \partial/\partial t$. Lighthill (1969) calculated the western boundary response in a manner similar to the above. That is, he also exploited the properties of the small ω large k modes to arrive at an equation like (14d) and a boundary condition like (18). However, he neglected the Kelvin mode part of the response and thus overestimated the northward mass flux in the Somali Current (Dennis Moore first pointed this out to us in 1972).

b. Eastern boundary response. The response to an incident zonal current of the form (10) (i.e., $u=u(y)t^s$) at an eastern boundary $x = X_E$ is an infinite sum

$$\mathbf{u}^E = \sum_{n=1}^{\infty} a_n H(\xi_n) \{\xi_n{}^s \mathbf{R}_n + s\mathbf{V}_n\} \text{ for } s=0, 1 \tag{20}$$

where $\xi_n = t + (2n+1)(x-X_E)$. Each term in the sum is a synthesis of long wave Rossby waves (cf. I, Sec. 6) and has u and h in geostrophic balance. The a_n's may be computed by applying Moore's algorithm in the long wave limit $k_n \approx -(2n+1)\omega$. If the incident motion has the form (15); then, with $\alpha_m{}^n$ given by (16)

$$a_n = 2(J+2)\alpha_{J+2}{}^n . \tag{21}$$

The most important example of an eastern boundary reflection occurs when the incident motion is a Kelvin mode: take its amplitude-time dependence to be $A\pi^{1/4}t^s$ so that in (20)

$$a_n = A \cdot 2\pi^{1/4}\alpha_1{}^n \,. \tag{22}$$

Now $(u,v,h) = A\pi^{1/4}t^s\mathbf{M}_{-1} + \mathbf{u}^E$ is the sum of the incident and reflected motions. At $x=X_E$ we have $u=v=0$ and $h=A$ for $s=0$ while $u=0$, $v=Ay$ and $h=At$ for $s=1$. These results follow either from considering the series (20) or from the following. That $u=0$ is of course the boundary condition at $x=X_E$. It then follows from geostrophy that $h_y=0$. Hence $h=f(t)$ with the exact form of $f(t)$ determined by the fact that the amplitude of the Kelvin component of h must be the projection of $f(t)$ on $2^{-1/2}\psi_0(y)$. That $v=Ay$ then follows directly from (17).

It will prove useful in Sec. 5 to note that the $s=0$ result leads us to the expansion

$$(0,0,A) = A\pi^{1/4}[\mathbf{M}_{-1}+2 \sum_{n \text{ odd}} \alpha_1{}^n\mathbf{R}_n] \,; \tag{23}$$

that is, the combination of modes on the right-hand side of (23) uniformly raises the sea level everywhere by an amount A. (It is the response to a constant impulse heat source; cf. Cane, 1974). This sum satisfies the boundary conditions since it gives $u=0$ for all y. We have already seen that the Rossby modes in (22) are the eastern reflection of the Kelvin mode. It may also be shown that the Kelvin mode is the western boundary reflection of all the Rossby modes. (The proof makes use of Eq. (A2) as well as (16). Using (A3) one may also verify that there is no trapped western boundary response).

Moore (1968, Ch. 4) has shown that the reflection of an equatorial Kelvin wave asymptotes with increasing y to a coastal Kelvin wave. Since a coastal Kelvin wave is *trapped* to the coast within the local radius of deformation, this appears to conflict with our result that the Kelvin mode reflection is a series of westward *propagating* Rossby modes. To clarify the situation, consider an incident Kelvin wave at a low but nonzero frequency ω. The reflection will consist of an infinite series of Rossby waves at the same frequency. It then follows from (2a) that there is a $N_0\,(\omega)$ such that the nth Rossby wave propagates if $n<N_0$ but is trapped if $n>N_0$. It further follows from the properties of the Hermite functions that the nth wave has exponentially decaying amplitude at latitudes beyond the turning latitude $Y_T(n) = (2n+1)^{1/2}$. Hence only coastally trapped Rossby modes are substantially present at latitudes $y>>Y_T(N_0)$; the coastal Kelvin wave is comprised of such modes.

Consider now a latitude $y = (2M+1)^{1/2}$ with $M>>N_0$. Modes with $n<<M$ are exponentially small at latitude y while those with $n>>M$ are more tightly trapped to the coast so that the Mth mode is the widest one present at latitude y. From (2a) the width of this mode is given by

$$(Imk_M)^{-1} \approx (2M+1)^{-1/2} = y^{-1} \,.$$

Dimensionally, the last term is just $f(y)/(gH)^{1/2}$—that is, the local radius of deformation. The argument of this paragraph thus shows heuristically how the trapped Rossby modes combine to give the coastal Kelvin wave its shape.

The time dependences being considered in this paper have, asymptotically in time, an effective frequency $\omega=0$. From (2a) it is evident that as $\omega \to 0$, $N_0 \to \infty$ so that more of the reflecting modes become propagating modes. Moore's analysis showed the coastal Kelvin wave is poleward of the latitude $Y_T(N_0)$: As $\omega \to 0$, $Y_T \to \infty$ so that the reflection consists of propagating modes everywhere. Our results are consistent with Moore's analysis in this limit.

Anderson and Rowlands (1976a) interpret the asymptotic eastern boundary response to a step function equatorial Kelvin wave as a coastal Kelvin wave. A number of short time numerical simulations of oceanic response to a switched-on forcing (O'Brien and Hurlburt, 1975; Hurlburt *et al.*, 1976; Cane, 1975, 1976) appear to show a Kelvin wave propagating poleward along the eastern boundary. This may be explained as follows. At a latitude y_M the widest mode present will be the one for which $y_m^2 = 2M+1$: lower n modes have no amplitude at this latitude while larger n modes travel more slowly and so do not extend as far to the west. This mode has group velocity $-(2M+1)^{-1}$ so at time t it extends a distance $x_M = (2M+1)^{-1}t$. Now if we move up the boundary from the equator with Kelvin wave speed $c=1$, we arrive at latitude y_M at time $t=y_M$ at which time $x_M=y_M^{-1}$—the local radius of deformation. Thus if we move up the boundary at the Kelvin wave velocity, we always see the wave front at the local radius of deformation. The response thus has some of the characteristics of a Kelvin wave though no true Kelvin wave is present. Longer time integrations show that the reflection does in fact continue to propagate farther westward into the basin (Cane, 1975, 1976).

4. Spin-up in an equatorial ocean basin: general considerations

With some simple algebra we are now able to calculate the response of an equatorial ocean with meridional boundaries to an x-independent steady forcing switched on at $t=0$ (i.e. we can solve Eqs. 1a-e). The unbounded response is given by (3)*ff*. Exclusive of inertia-gravity waves the western boundary correction needed to bring the zonal velocity to zero may then be calculated by using (12) and (16); the eastern response has the form (20) with the coefficients determined by (21). The inertia-gravity part of the unbounded response is made up of waves at discrete frequencies and Moore's algorithm applies directly. Before we proceed to the details of some individual cases it is useful to provide a general framework.

a. Inertia-gravity waves. All of the inertia-gravity wave terms [Eq. (7)] have essentially the same form: their east-west wavenumber $k=0$ and their group velocity is to the east; the nth such mode is a linear combination of waves of frequency $\pm(2n+1)^{1/2}$. Its reflections are diagrammed in Figure 3 where the initial amplitude $= 1$. At a western boundary the response to each such wave is a similar wave with equal amplitude but exactly out of phase. The effect is a cancellation of the original wave which propagates away from the boundary with the group velocity of the

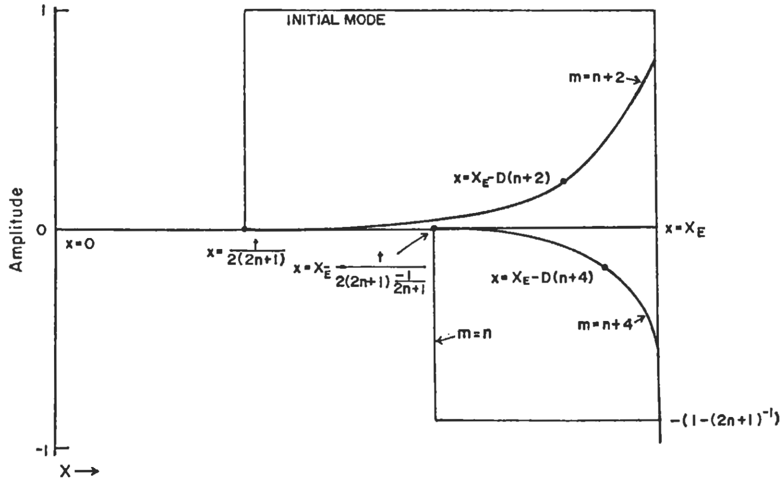


Figure 3. Amplitude of the inertia-gravity wave of index n and its boundary responses at time t. The effect of the western boundary is to just cancel the initial mode for $x \leqslant t[2(2n+1)]^{-1}$. The eastern boundary reflection is a series of modes with index $m=n$, $n+2$, $n+4, \ldots$. Only the $m=n$ mode propagates; the others are trapped with an e-folding scale given by $D(m) = [2(m-n) - 1/4(2n+1)^{-1}]^{1/2}$.

wave. [This response is exactly like the step function case, I, Eqs. (58b,c)]. The initial waves all have eastward group velocity and do not propagate energy toward a western boundary. This boundary response is not a reflection. Rather, the presence of a western boundary at $x=0$ makes the forcing function into a step function at the boundary, thus cutting off the energy source for motions which would otherwise propagate into the basin from the region west of $x=0$.

On the other hand, these $k=0$ inertia-gravity waves are carrying energy into an eastern boundary. The response must be motions which carry this energy away from the boundary. For the wave with meridional index n the largest fraction ($\approx 1-2(2n+1)^{-1}$) of this incident energy is reflected in a long ($k=(2n+1)^{-1}$) westward propagating wave with the same meridional index and the same frequency. The remaining energy is partitioned among an infinite set of modes with the same frequency and index $m>n$ and $m \equiv n$ mod 2. All of these modes are trapped at the eastern boundary since [cf. (2a) with $\omega^2=2n+1$]

$$k_m = -\tfrac{1}{2}(2n+1)^{-1/2} - i[2(m-n) - \tfrac{1}{4}(2n+1)^{-1}]^{1/2}\,.$$

Note that the reflection of the mixed mode ($n=0$) wave with $k=0$, $\omega=1$ is entirely in terms of trapped modes; there are no westward propagating waves at this frequency.

The nth westward propagating wave crosses the basin to the western side where it is reflected as a series of eastward propagating waves of index $m \leqslant n$ and $m \equiv n$

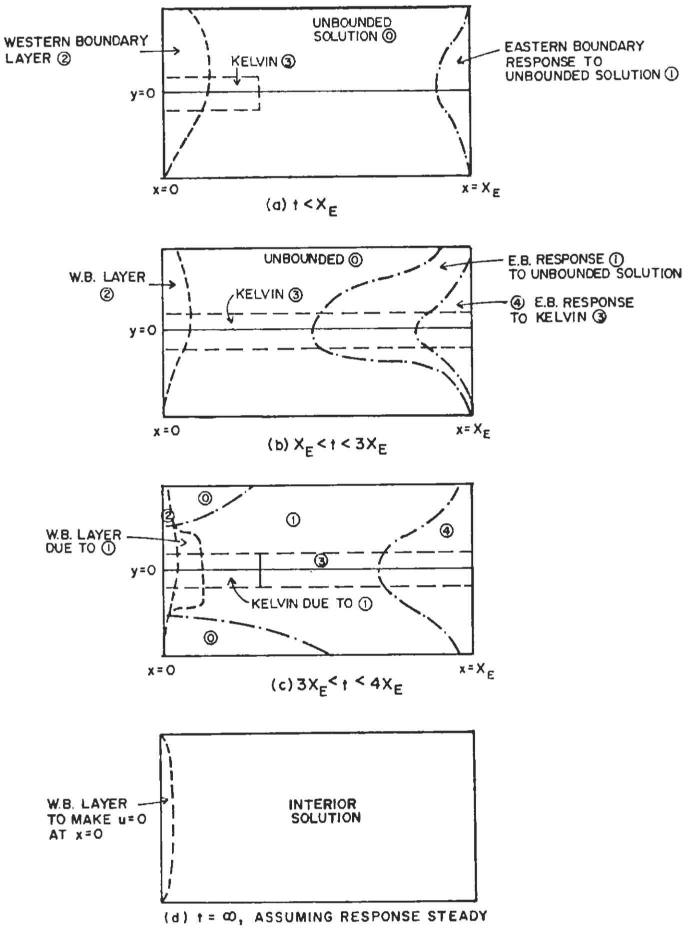


Figure 4. Sketch of the dominant nonoscillatory part of the baroclinic response in a bounded ocean, illustrating the component parts of the solution; see text.

mod 2. Most of the incident energy is reflected in the $m=n$, $k=0$ mode; this has the same structure as the initial eastward traveling wave.

The fate of the inertia-gravity waves may now be summarized as follows: when the forcing functions are turned on at $t=0$ eastward propagating $k=0$ inertia-gravity waves are generated. A region free of such waves expands eastward from the western boundary with time. (If the forcing were to extend only as far to the west as $x=a$ the oscillation free region is initially present to the west of $x=a$ and expands eastward from $x=a$). At the eastern side most of the energy of these initially generated inertia-gravity waves is reflected in the form of long wave westward travelling inertia-gravity waves. Some of the energy remains trapped at the eastern boundary. The westward propagating waves are reflected as a series of waves at the western boundary with most of their energy going into $k=0$ waves with the same structure as those initially generated. The net effect is that the initial waves are reflected back and forth across the basin. At each reflection at the eastern side there is a gradual loss of energy to oscillatory boundary trapped motions. It is obvious that the inertia-gravity waves do not bring the circulation toward a steady state. Their principal role in the spin-up of the ocean relates to the establishment of the steady motions in the unbounded solution by providing for the initial adjustment from the resting initial conditions (see the discussion in I. Sec. 5). They play no role in making the nonoscillatory motions "feel" the presence of boundaries. For this reason the inertia-gravity waves will not be discussed further.

b. Nonoscillatory motions. Figure 4 provides a schematic view of the non-inertia-gravity wave components of the solution. The zonal current described by the unbounded response (3) is generally nonzero so that additional motions are stimulated by the presence of boundaries. At the eastern side the additional motions are syntheses of long wave Rossby waves ($\omega \approx k \approx 0$) as described by (20). These modes propagate relatively rapidly: the group velocity of the nth mode is $-(2n+1)^{-1}$. Since the more equatorially confined lower n modes propagate faster, this response extends further into the basin near the equator and becomes narrower with increasing latitude. Only the first N modes travel fast enough to have reached longitude x by time t where $N(x,t)$ is the largest integer such that

$$2N+1 \leqslant t(X_E-x)^{-1} . \tag{24}$$

The resulting bulge in the eastern boundary reflection is illustrated in Figure 4 with a dotted line indicating the wave front marking its farthest westward extent. Note that at time t the front can travel no further west than to $x = X_E-t/3$.

The western boundary response (12) consists of a Kelvin mode travelling away from the boundary with group velocity 1 and a boundary trapped part (13) that grows narrower and more intense with time as indicated in Figure 4. The latter is a sum of modes that are a synthesis of short wavelength Rossby waves with low group velocity so that these modes stay near the western boundary. Most of their energy is in the v *component,* which is in geostrophic balance. Since their group velocity is so low, their energy density must be high in order for their energy flux to balance that of the incident motion. These features are qualitatively similar to the midlatitude case.

The asymmetry in the character of the eastward and westward propagating Rossby waves helps to explain why currents intensify on the western side of the ocean (Pedlosky, 1965). In addition, this reflection has features which are distinctly equatorial. Specifically, each incoming wave reflects as a whole series of waves, including the mixed mode or the Kelvin wave. Since the Kelvin waves carry energy away from the western boundary quickly, less of the incoming energy flux remains in the western boundary current than is the case for midlatitudes.

At $t=X_E$ the Kelvin mode from the western boundary arrives at the east and is reflected as a new series of longwave Rossby modes; see Figure 4b. By $t=3X_E$ the initial eastern boundary reflection has crossed the basin and stimulates a new Kelvin wave as well as additional boundary trapped motions. The significant difference from midlatitudes is the existence of signals that can traverse the basin rapidly.

c. Steady state solutions. We will be particularly interested in how the time evolving circulation approaches a steady state. Even though the forcing is steady these inviscid equations (1) need not reach a steady state when started from a resting initial state. Nevertheless, we anticipate that the long time circulation will bear some special relation to the steady circulation, perhaps, for example, oscillating about it. The steady inviscid equations generally do not admit solutions satisfying $u=0$ at both $x=0$ and $x=X_E$. It is well known that the addition of viscosity permits a viscous boundary layer at the western side only. Furthermore, as illustrated in Figure 4d, the retention of the time dependences in (1) permits a steady state flow to be corrected by a time-dependent boundary layer at the western side. (Cf. (13) with $s=0$). We therefore envision a "steady state" solution to (1) as actually consisting of a steady interior solution plus a time dependent boundary layer correction at the western side. Hence we follow Sverdrup (1947) and impose the condition $u=0$ at the eastern side $x=X_E$ on the interior solution. Applying this to the steady form of (1) yields

$$u = -\int_{X_E}^{x} [G_x - F_y]_y dx + \int_{X_E}^{x} [yQ_y + 2Q] dx \tag{25a}$$

$$v = [G_x - F_y] - yQ \tag{25b}$$

$$h = \int_{X_E}^{x} \{y[G_x - F_y] + F\} dx + \int_{-\infty}^{y} G(x=X_E) dy - \int_{X_E}^{x} y^2 Q dx + h_0 \tag{25c}$$

where h_0 is independent of x and y (see below).

If $Q\equiv 0$ then the circulation is purely wind-driven in which case (25) reduces to Sverdrup's (1947) solution. If the curl of the wind stress is zero then there is no steady motion and the sea surface setup balances the wind stress: $h_x=F$, $h_y=G$. Such a solution satisfies all boundary conditions without the need of a western boundary layer. For a thermally driven circulation ($F\equiv G\equiv 0$, $Q\neq 0$) (25) says that the steady solution is geostrophic with the thermal source locally balanced by mass divergence.

The constant h_0 appearing in (25c) is determined by a mass conservation condition. Consider first the case where the integral of Q over the basin is zero so that the amount of mass in the basin is not altered by external sources. Then the appropriate condition is that the integral of h over the basin vanishes so that mass is conserved. The contribution from the boundary correction h^B must be included. Since h^B is given by (12) with $s=0$, it follows that

$$\int_0^{X_E} h^B(x,y,t)dx \cong \int_0^{x=ct} h^B(y)J_0\,(2\sqrt{xt}) = (x/t)^{1/2}J_1\,(2\sqrt{xt})h^B(y) \tag{26}$$

where c is the maximum eastward group velocity. Hence as $t \rightarrow \infty$ it follows that $\int_{-\infty}^{+\infty} dy \int_0^{X_E} dx\; h^B = 0(t^{-1/2})$ so that the mass contained in the boundary layer is negligible compared to that in the interior. (The same conclusion follows for a viscous boundary layer).

A second problem in determining h_0 is somewhat artificial and arises from taking the basin to extend infinitely to the north and south. If the basin had zonal walls at $y=Y_N$ and $y=Y_S$ then

$$h_0 = -[X_E(Y_N-Y_S)]^{-1} \int_{Y_S}^{Y_N} \int_0^{X_E} h'dxdy \tag{27}$$

where $h' = h-h_0$ as defined by (25c). For an infinite ocean the condition that $\int\int hdxdy$ vanish is less compelling but still applies if no mass is supplied at $y = \pm\,\infty$. Then h_0 is defined by (27) in the limit as $Y_N, -Y_S \rightarrow \infty$.

If $\int\int Qdxdy \equiv \overline{Q} \neq 0$ then the mass of the basin is continually increased at a rate $\overline{Q}$ so that mass conservation requires that $\int\int hdxdy = \overline{Q}t$. Eq. (26) shows that the western boundary layer cannot absorb the additional mass so the steadiest possible interior solution is now given by (25) with h_0 equal to (27) less $\overline{Q}t[X_E(Y_N-Y_S)]^{-1}$. All motions are steady but the depth of the ocean rises steadily. The simplest example is provided by the response to $Q=1$. In this case $u=v=0$ and $h=t$.

5. Spin-up in an equatorial ocean basin: results

In this section we will consider in some detail the response of a bounded equatorial ocean to an imposed wind stress. The discussion will focus on the approach of the time dependent circulation to the steady state solution (25). We will be concerned with whether or not (25) can be attained from a resting initial state without invoking effects left out of Eqs. (1) (i.e., friction, nonlinearity, effects of zonal walls). As mentioned in the previous section we shall not elaborate further on the inertia-gravity wave motions.

Since the equations are linear the response to a general wind stress $(F(y), G(y))$ will be a linear combination of the responses to $(0, G_S(y))$, $(F_A(y), 0)$, $(0, G_A(y))$ and $(F_S(y), 0)$ where subscript S denotes the part of the function that is symmetric about the equator and A the antisymmetric part (e.g., $F_S(y) = 1/2[F(y) + F(-y)]$,

$F_A(y) = 1/2[F(y) - F(-y)]$). The four cases will be considered in order of increasing complexity after which the response to $Q \neq 0$ will be briefly discussed.

a. G(y) symmetric about the equator. The ocean circulation will have u and h antisymmetric about the equator and v symmetric. Hence it will be made up of modes for which n is even so that no Kelvin waves will appear. The steady solution has

$$u = v = 0, h_y = G. \tag{28}$$

Exclusive of inertia-gravity waves the solution in the absence of boundaries takes the form $u = u^{(2)}(y)$, $h = h^{(2)}(y)$ and $v = 0$ [cf. (3)] so that (1b) becomes

$$yu^{(2)} + h_y{}^{(2)} = G. \tag{29a}$$

The eastern boundary response $\mathbf{u}^E$ takes the form (20) with $s = 0$ (i.e. u^E and h^E are independent of t and $v^E = 0$). It is a sum of long wave Rossby modes that carry energy westward. Since each term of the sum (20) has $v = 0$ and u and h in geostrophic balance it follows that

$$v^E = 0, yu^E + h^E{}_y = 0. \tag{29b}$$

A mode indexed by n has westward group velocity of magnitude $(2n + 1)^{-1}$; hence, for a given x and t

$$(u^E(x,y,t),0,h^E(x,y,t)) = \sum_{n=1}^{N(x,t)} a_n'\mathbf{R}_n(y) \tag{30}$$

where N is given by (24). This simply says that the solution at a point (x, t) consists only of those modes which propagate fast enough to have reached x from the eastern boundary. Since the group velocity decreases with increasing n, and since the modes with smaller n are more equatorially confined, for a given distance from the eastern boundary the response is felt more quickly the closer one is to the equator. (This is shown schematically in Figures 4 and 5). For $G(y)$ symmetric about the equator $a'_n = 0$ for n odd so that the lowest mode present has $n = 2$ and the boundary response extends no further to the west than $x = X_E - t/5$.

Right at the eastern wall all modes are present; adding (29a) and (29b) at x=X_E

$$y(u^{(2)} + u^E) + (h^{(2)} + h^E)_y = G.$$

Since u^E at X_E is determined by the boundary condition that $u^E = -u^{(2)}$, at x=X_E

$$u^E + u^{(2)} = 0; v^E + v^{(2)} = 0; (h^E + h^{(2)})_y = G. \tag{31a}$$

This is precisely the steady solution (28). Since each mode of h is antisymmetric the total h is so that at each longitude $\int_{-\infty}^{+\infty} hdy = 0$. For a point away from the wall

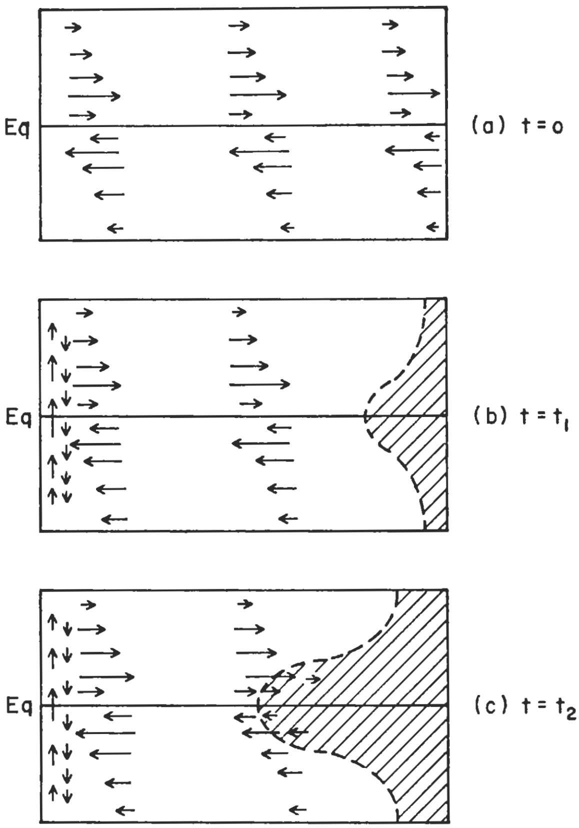


Figure 5. Sketch of the evolution of the flow in response to a meridional wind stress $G(y)$ with even symmetry about the equator.

$$(u^{(2)} + u^E, v^{(2)} + v^E, h^{(2)} + h^E) = (0, 0, \textstyle\int G dy + h_0) - \sum_{n=N+1}^{\infty} a'_n \mathbf{R}_n \tag{31b}$$

so that the last sum gives the deviation from a state of no motion with the wind stress balanced by the tilt of the height field. For a fixed x, N increases as time passes—more and more modes arrive at the longitude x—so the balanced state is approached more closely. Thus the influence of the eastern boundary brings the ocean toward the steady state Sverdrup balance (in this case, no motion with the wind stress balanced by sea surface setup). By the time $t=5X_E$ the $n=2$ mode has reached the western boundary bringing the region very near the equator (i.e. within about a radius of deformation) close to a steady state at all longitudes. At $t=9X_E$, $13X_E$, $17X_E$, etc. the $n=4,6,8$, etc. modes arrive at $x=0$. Each mode brings the low latitudes still closer to the final state while extending the adjusted region poleward. Obviously, midlatitudes take much longer than equatorial latitudes to reach a final state. (At 30° the major contribution to the response would be the first baroclinic mode with $n \approx 50$; these would not cross the basin until $t \approx 100 X_E$.

There may be some initial puzzlement when one first considers the mass fluxes that go with the solution outlined above. For definiteness in this discussion, let us assume the wind is everywhere from the south. Then the adjustment which is to be reached requires that mass be moved from the southern to the northern hemisphere, yet the modes which apparently do the adjusting have no (asymptotically large) north-south velocities associated with them. The mass flow may be described as follows: when the inertia-gravity waves which are initially excited are cleared away, there remains a steady flow toward the eastern wall north of the equator and away from the eastern wall south of it. As the front which marks the edge of the eastern boundary response (the dashed line of Figure 5) moves away from the wall, it leaves behind a region where the zonal velocity is reduced in magnitude. Hence, there is a convergence of mass into this region north of the equator and a divergence out of it south of the equator. If there were no western boundary, this process would simply roll on toward $x = -\infty$. The presence of a western boundary makes it necessary for the mass flowing westward in the southern hemisphere to be carried northward across the equator in a western boundary current. It then flows eastward to pile up behind the front advancing from the east (Fig. 5).

This western boundary current is the boundary layer solution $\mathbf{u}^B$ of Eq. (13). Since only modes with n even are present v^B is symmetric and hence able to supply the required cross-equatorial transport. Also, no Kelvin waves are excited, consistent with the fact that for this symmetry there is no net mass flux into the western boundary. The role of the western boundary layer is solely to redistribute the incoming mass meridionally; note that, with u^I as the interior zonal velocity at $x=0$

$$\int_0^\infty v_y{}^B dx = -\int_0^\infty u_x{}^B dx = u^B(x=0) = -u^I\,;$$

which says that at each latitude the incoming (outgoing) zonal mass flux of the interior solution goes entirely to increase (decrease) the meridional mass flux of the boundary current. Initially $u^I = u^{(2)}$; as more and more modes arrive from the east, u^I tends toward zero, thereby bringing $\mathbf{u}^B$ toward zero.

b. F(y) antisymmetric about the equator. The symmetries in the resulting circulation are the same as in the previous case. Only even n modes are present and there are no Kelvin modes. The steady solution is

$$u = (x-X_E)F_{yy},\ v = -F_y,\ h = (x-X_E)\,(F-yF_y)\,. \tag{32}$$

Since the curl of the wind stress need not vanish, there may be nonzero currents associated with this circulation.

The unbounded solution takes the form $\mathbf{u}^{(1)}$ specified by (3)*ff*. Eq. (4) shows that the secularly growing part is a sum of Rossby modes. As was discussed in I, Sec. 4, this growth comes about as the difference between a freely propagating motion (of the form (10) with $s=1$) and a locally forced response. The propagating modes are required to satisfy the initial conditions. Since they propagate energy westward, the eastern boundary does not act on these modes as a reflecting barrier. Rather, it acts to cut off the forcing to the east of $x = X_E$. The same effect is evident in the barotropic example of Sec. 1. (Also see the discussion of inertia-gravity waves at a western boundary in Sec. 4).

The eastern boundary thus has the effect of removing the propagating Rossby modes from the unbounded solution. The response $\mathbf{u}^E$ is given by (20) with $s = 1$ and $a_n = -r_n$. At $x = X_E$, where all modes are present

$$v = v^{(2)}(y) + v^E(y) = -\sum_{n=0}^{\infty} (2n+1)^{-1} d_n \psi_n - \sum_{n=0}^{\infty} [e_n - (2n+1)^{-1} d_n]\psi_n ;$$

$$v = -\sum_{n=0}^{\infty} e_n \psi_n = -\sum_{n=0}^{\infty} (F_y)_n \psi_n = -F_y . \qquad (33a)$$

Similarly, at $x=X_E$

$$(u,0,h) = (tu^{(1)} + u^E, 0, th^{(1)} + h^E) = -(x-X_E) \sum_{n=1}^{\infty} (2n+1) r_n \mathbf{R}_n . \qquad (33b)$$

The value of v given by (33a) is the same as that for the steady state solution (32). It is evident from (33b) that u and h have the same form as their steady state counterparts; i.e., $(x-X_E)$ times functions of y only. That these functions are in fact the same as the ones in (32) may be verified by writing the sum in (33b) in the form $\sum_{n=1}^{\infty} (u_n, 0, h_n)\psi_n$ and checking directly that $u_n = (F_{yy})_n$ and $h_n = (F - yF_y)_n$. More easily, one may note that (33) must satisfy the steady state form of (1) so that (33) and (32) must be the same.

For a longitude x away from the eastern boundary, the circulation has the form (32) less the sum of modes that have not yet arrived at x. This is similar to the solution (31b) for the previous case. As time passes, the influence of the eastern boundary is felt for the more slowly travelling high n modes so that the circulation at a point (x, y) approaches the steady state circulation (32) more closely. Thus, as in the case of a symmetric meridional wind, the presence of an eastern boundary brings the ocean toward the steady state Sverdrup balance. In the present case, the curl of the wind stress may be nonzero showing that the way in which the spin-up takes place is not governed by the presence or absence of wind stress curl. In this response to a zonal wind, a significant part of the mass redistribution required to reach the final state is accomplished by meridional currents.

The western boundary correction has the form (13); it contains no Kelvin modes and is entirely boundary trapped. [It may have a mixed mode component]. If the steady interior u is nonzero at this boundary, a boundary layer must be present for all time in order to redirect the incoming mass fluxes and thus complete the fluid circuits established by the interior circulation. This contrasts with case a.

Since the steady circulation (32) has motions in the direction of the wind, there is the possibility of a constant net input of energy by the wind stress. In fact,

$$E \equiv \int_{-\infty}^{+\infty} \int_{0}^{X_E} uF\,dxdy = -\tfrac{1}{2} X_E^2 \int_{-\infty}^{+\infty} FF_{yy}dy = \tfrac{1}{2} X_E^2 \int_{-\infty}^{+\infty} (F_y)^2 dy$$

If the interior solutions are steady, this energy input must be absorbed in the western boundary layer. If this layer is viscous, the energy will be dissipated, but with the inviscid equations the rate of increase of energy in the boundary layer must be equal to E. To highest order (in t), all of the boundary layer energy is in the meridional velocity component. Now,

$$u^B(x{=}0) = -u(x{=}0) = X_E F_{yy}$$

so that it follows from (13) that

$$v^B(x, y, t) = -X_E F_y(y)\, (t/x)^{1/2} J_1\,(2\sqrt{xt})\,.$$

Then

$$\partial/\partial t \int_{-\infty}^{+\infty} \int_{0}^{X_E} \tfrac{1}{2} (v^B)^2 dxdy = \tfrac{1}{2} X_E^2 \int_{-\infty}^{+\infty} F_y^2 dy = E$$

as is required. In addition the wind puts in vorticity at a rate $-X_E F_y(y)$ at latitude y; this is precisely the rate of increase of the vorticity of the meridional boundary current v^B at this latitude. While the boundary layer is unable to absorb mass [Eq. (26)] it is able to absorb the energy and vorticity put into the ocean by the wind so the interior circulation can be steady.

c. G(y) antisymmetric about the equator. The ocean circulation will have u and h symmetric about the equator and v antisymmetric. It consists of modes for which n is odd so that Kelvin waves may appear. Note that the unbounded response has no Kelvin mode; see (6). The steady oceanic response is the same as that to a symmetric meridional wind. It consists of a state of no motion with the wind stress balanced by the sea surface setup; viz. (28).

In fact, in many respects the evolution of the circulation for G antisymmetric is the same as that for G symmetric. All of the arguments leading to (31a) still apply: the unbounded response satisfies (29a) and $v{=}0$. The eastern boundary reflection again consists of long wave Rossby modes satisfying (29b) so that (28) is satisfied at the eastern boundary and approached more and more closely at longitudes away from the boundary as time passes and more and more eastern boundary modes reach a given longitude. [For this symmetry the $n=1$ mode may be present in which case the eastern front will extend as far to the west as $x=X_E-t/3$].

However, in going from (31a) to (31b) we made use of the antisymmetry of h to conclude that the mass conservation condition was satisfied. In the present case it is again true that

$$\int_{-\infty}^{+\infty} h^{(2)}\,dy = -\int_{-\infty}^{+\infty} dy \Sigma g_n (2n+1)^{-1} d\psi_n/dy = -\Sigma\, g_n (2n+1)^{-1} \int_{-\infty}^{+\infty} dy\, d\psi_n/dy = 0\,,$$

so that there is no net mass associated with the unbounded part of the solution. However, at $x=X_E$

$$\int_{-\infty}^{+\infty} h^E dy = \int_{-\infty}^{+\infty} \Sigma a_n (2n+1)\,[4n(n+1)]^{-1}[y\psi_n + (2n+1)^{-1}\, d\psi_n/dy]dy\,,$$

$$\int_{-\infty}^{+\infty} h^E dy = \sum_{n=1}^{\infty} a_n (2n+1)\,[4n(n+1)]^{-1} \int_{-\infty}^{+\infty} y\psi_n dy$$

$$= 2\pi^{1/4} \sum_{n=1}^{\infty} a_n\,(2n+1)\,[4n(n+1)]^{-1}\alpha_1{}^n \tag{34a}$$

where

$$a_n = 2g_n \bullet n\,(2n+1)^{-1} + 2 \sum_{J=1}^{n-2} g_J \alpha_J{}^n\,. \tag{34b}$$

In general $\int h^E(x=X_E)dy \neq 0$ so that although the solution $\mathbf{u}^{(2)} + \mathbf{u}^E$ with all modes present satisfies (28) (i.e. it gives the correct sea surface tilt) it does not satisfy the proper mass conservation condition. Furthermore, the mass is not simply redistributed behind the eastern front; it follows from (34) and (A4) that (for $t<3X_E$)

$$I^E = \int_0^{X_E} dx \int_{-\infty}^{+\infty} dy\{h^E\}$$

$$= 2\pi^{1/4} \sum_{n=1}^{\infty} \int_0^{X_E} dx H[t+(2n+1)\,(x-X_E)]\; a_n \alpha_1{}^n (2n+1)\,[4n(n+1)]^{-1}$$

$$= \tfrac{1}{2}\pi^{1/4} t \sum_{n=1}^{\infty} a_n \alpha_1{}^n [n(n+1)]^{-1}$$

$$= \pi^{1/4} t \sum_{J=1}^{\infty} g_J \{\alpha_1{}^J[(J+1)\,(2J+1)]^{-1} + \sum_{n>J} \alpha_1{}^n \alpha_J{}^n [n(n+1)]^{-1}\,;$$

$$I^E = 2\pi^{1/4} t \sum_{J=1}^{\infty} g_J (2J+1)^{-1} \alpha_1{}^J\,. \tag{35}$$

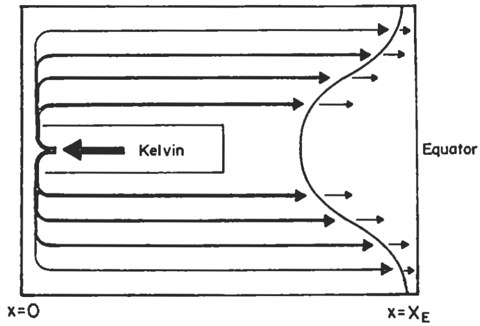


Figure 6. Sketch of the mass flux at a time $t<X_E$ in response to a meridional wind stress $G(y)$ that is everywhere poleward.

In general $I^E \neq 0$ so there is a net change in the mean height behind the eastern front. Even if $\int h^E dy$ should be zero at the boundary I^E will be nonzero (except for wind stresses with very special meridional structures). This is a consequence of the fact that the modes coming off the eastern boundary travel at different speeds: if they all propagated at the same rate I^E would be a multiple of $\int h^E$.

For definiteness in the discussion let us assume that there is a mass surplus at the eastern boundary and that $G(y)$ is poleward everywhere. Since mass is conserved within the ocean basin in the absence of sources at infinity there must be a region with a mass deficit to offset the surplus at the eastern side. This region is (initially) localized at the western boundary. The western boundary response to the unbounded solution has the steady form (13) with $s=0$; we have already seen (Eq. 26) that the mass contained in the boundary trapped part of this response is asymptotically small. Since there will generally be a net mass flux into or out of the boundary (e.g., for $G(y)$ poleward $u^{(2)}$ is everywhere eastward) a Kelvin mode will also be generated at the boundary. At time t the Kelvin mode has propagated to $x=t$; denoting its height by h^K and calculating its amplitude b_{-1} from (16) we obtain (for $t<X_E$)

$$\int_0^{X_E} dx \int_{-\infty}^{+\infty} dy\{h^K\} = b_{-1}t \int_{-\infty}^{+\infty} 2^{-1/2}\psi_0 dy = \pi^{1/4} t b_{-1}$$

$$= -2\pi^{1/4} t \sum_{J=1}^{\infty} g_J(2J+1)^{-1}\alpha_1{}^J = -I^E .$$

The mass deficit associated with the Kelvin mode is equal in magnitude to the surplus behind the eastern front. The region where the Kelvin mode is present is thus the source for the extra mass at the east. Figure 6 shows the mass fluxes for the illustrative example with G poleward everywhere. The eastward Kelvin mode provides a westward current that carries mass into the western boundary. The bound-

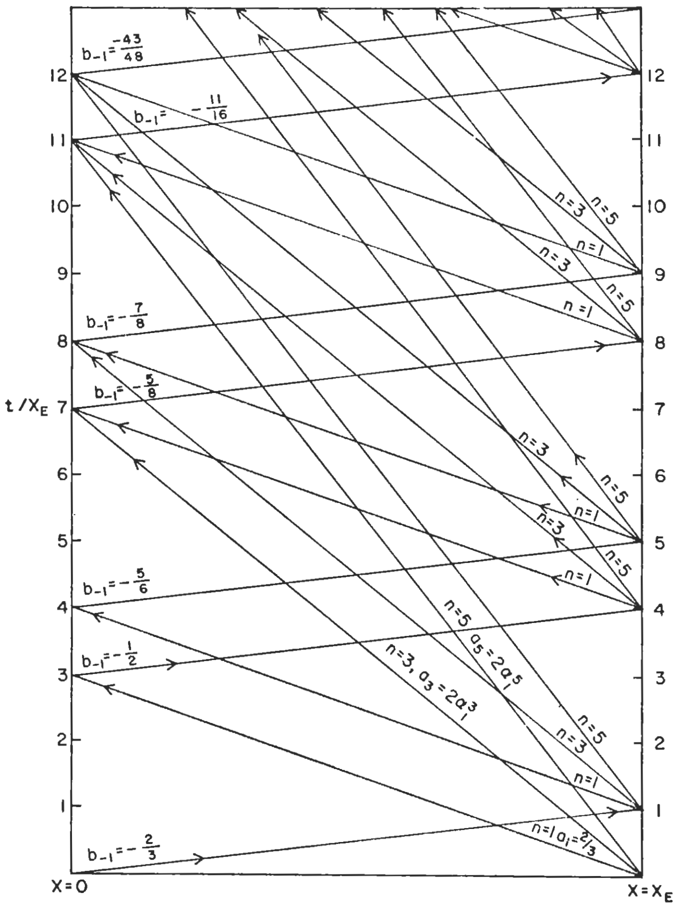
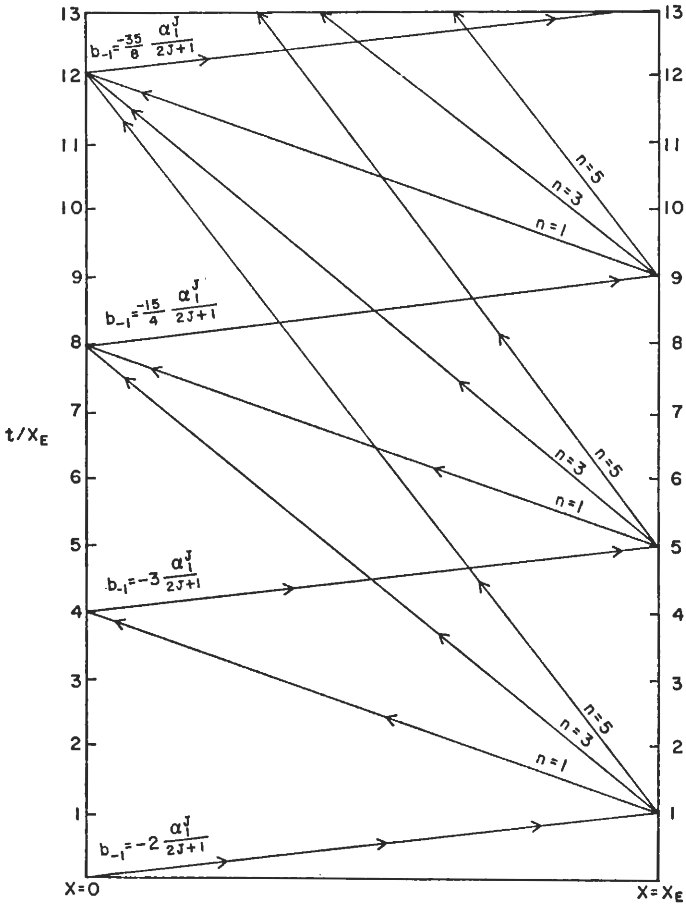


Figure 7. Diagram of the reflections that take place as the flow evolves. It shows the Kelvin mode amplitude b_{-1} at the western boundary and the Rossby modes generated at the eastern boundary for $n \leqslant 5$. (a) The response to an equatorially confined meridional wind stress $G(y) = \psi_1(y)$. (b) The response to a midlatitude wind stress $G(y) = \psi_J(y)$, $J >> 1$.

ary current distributes this mass meridionally and it is carried eastward by the interior flow until the eastern front is reached. The zonal velocity behind the front is less eastward, so there is a convergence of mass into this region. As a result there is a region of lowered height along the equator extending from $x=0$ to $x=t$ and a region of raised height along the eastern boundary.

The preceding paragraph describes the situation until time $t=X_E$ when the Kelvin mode arrives at the eastern boundary. As we have already seen [Eq. (23)] the Kelvin mode plus its reflection sum to $u=v=0$ and h=constant. This leaves the balance (28) unaltered. Since a mass excess behind the original eastern front meant the Kelvin mode carried a mass deficit, the arrival of the Kelvin mode tends to



reduce the excess at the eastern boundary. The effect is to bring the circulation closer to the steady state solution, though the height at the eastern boundary does not yet take on the correct value.

One way to reach this conclusion is to note that the Kelvin mode removes mass from the region behind the eastern front at the same rate that it is being added (by the motions associated with the unbounded solutions). Examination of the series shows that the reflection of the Kelvin mode has more amplitude than u^E in the low n modes that are more equatorially confined. Since these propagate more rapidly, the Kelvin reflection spreads more of the mass deficit over a greater longitudinal distance than u^E does the surplus. As a result there is a continued mass excess near the boundary at higher latitudes.

To summarize the results to this point: the initial unbounded solution together with the eastern boundary response tend to satisfy the steady state balance (28) but the mean sea level is generally not correct; e.g. it is too high. The source of

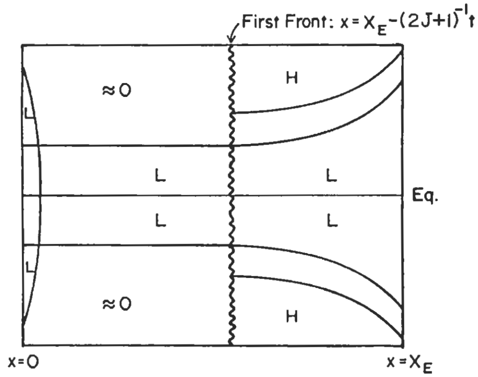


Figure 8. Schematic illustration of the contours of h in response to $G(y) = \psi_J(y)$, $J>>1$ at a time $1<<t/X_E<2J+1$.

this extra mass is a Kelvin mode originating at the western boundary. At $t=X_E$ this Kelvin mode reaches $x=X_E$ where it, together with its reflection, tends to reduce the mean sea level uniformly in y. Fig. 7a diagrams the evolution of the flow for $G(y) = \psi_1(y)$. At $t=3X_E$ the $n=1$ mode of the eastern boundary reflection arrives at the west and is reflected as a new Kelvin mode of amplitude 1/6. The total amplitude of the Kelvin mode leaving the western boundary is thus reduced from −2/3 to −1/2. At $t=4X_E$ the $n=1$ mode generated as part of the eastern reflection of the first Kelvin mode reaches the western boundary stimulating a new Kelvin mode and changing the amplitude to −5/6—more negative than the initial value of −2/3. Note the way the Kelvin amplitude oscillates about its initial value. It becomes less negative when a new mode from the eastern boundary reflection u^E arrives to reduce the mass flux away from $x=0$. It becomes more negative when a mode generated as a Kelvin reflection at the east arrives to increase the mass flux. Speaking anthropomorphically, all of these motions are trying to bring the basin toward a steady state but they travel at different speeds or they started at different times and cannot get synchronized. A certain amount of mass sloshes back and forth as a result. As seen in Figure 7a the amplitude of these sloshings decreases slowly with time so the circulation does slowly approach the steady state. However, a steady state cannot actually be attained without considering other processes (e.g., nonlinearity, friction).

Figure 7b diagrams the evolution when $G(y) = \psi_J(y)$, $J>>1$; i.e. when G has most of its amplitude in midlatitudes. The eastern boundary reflection, being composed of modes with $n \geqslant J$, is also present primarily in midlatitudes. None of its modes reach the western side until $t=(2J+1)X_E$. In the interim the Kelvin modes and the low n components of their eastern boundary reflections have crossed the basin. The result is that while the sea level is raised near the eastern boundary at high latitudes it is lowered near the equator all across the basin. This is shown schematically in Figure 8.

d. F(y) symmetric about the equator. The oceanic response has the same symmetry as in case *c*, but in this case the unbounded solution contains a Kelvin mode as long as $(F)_0$ is nonzero [cf. (9a)].

No initial Kelvin mode. Let us begin by assuming that $(F)_0 = 0$ so the Kelvin mode is absent initially. If this is true then the response to $F(y)$ symmetric combines the properties of the responses studied in cases b and c. As in case b (33) again holds at the eastern boundary so that behind the front at the east the response approaches the steady form (32) (in the by now usual sense that at a longitude x the first $N(x,t)$ modes that comprise (32) have arrived by time t). However, with F symmetric (32) may not be the steady solution. The components of the current are correct but

$$\int_{-\infty}^{+\infty} dy \int_0^{X_E} dx\, h = -\tfrac{1}{2} X_E^2 \int_{-\infty}^{+\infty} dy(F - yF_y) = -X_E^2 \int_{-\infty}^{+\infty} F dy$$

and since the last integral need not be zero the requirement that mass be conserved may not be fulfilled. The situation is similar to case c in that the response behind the eastern front tends to the steady currents and the steady gradient of h but the mean level is not correct. Even if $\int_{-\infty}^{+\infty} F dy = 0$ so that the level is correct at the eastern boundary, there will usually be a mass deficit or excess in the region behind the eastern front. As discussed under case c this is due to the different propagation speeds of the components of the eastern boundary reflection.

For definiteness in the discussion let us say that there is an excess of mass at the east. Note that since the unbounded nonoscillatory solution $(u^{(1)}t,\ v^{(1)},\ h^{(1)}t)$ satisfies (1) it follows from (1c) that

$$\int_{-\infty}^{+\infty} h^{(1)} dy = -\int_{-\infty}^{+\infty} v_y^{(1)} dy = 0 \tag{36}$$

As in case c all of the mass excess is associated with the eastern boundary response h^E. As before, somewhere within the basin there must be a region with a mass deficit to offset this surplus. We again look at the western boundary. Since the zonal current incident on the western boundary is growing linearly in time the boundary response is given by (13) with $s=1$. In analogy to (26) one may readily show that the net mass contained in the boundary trapped part of the response is (at most) $0(t)$ while that in the Kelvin mode is $b_{-1}\pi^{1/4}t^2/2$ so that to leading order in t the deficit is all in the Kelvin mode. In fact

$$\int_{-\infty}^{+\infty} dy \int_0^{X_E} dx\, h^E = -\Sigma\, r_n t^2 [8n(n+1)]^{-1} \int_{-\infty}^{+\infty} y\psi_n dy =$$
$$-\tfrac{1}{2}\pi^{\frac{1}{4}} t^2 \Sigma\, r_n \alpha_1{}^n [2n(n+1)]^{-1} = -\tfrac{1}{2}\pi^{\frac{1}{4}} t^2 b_{-1}$$

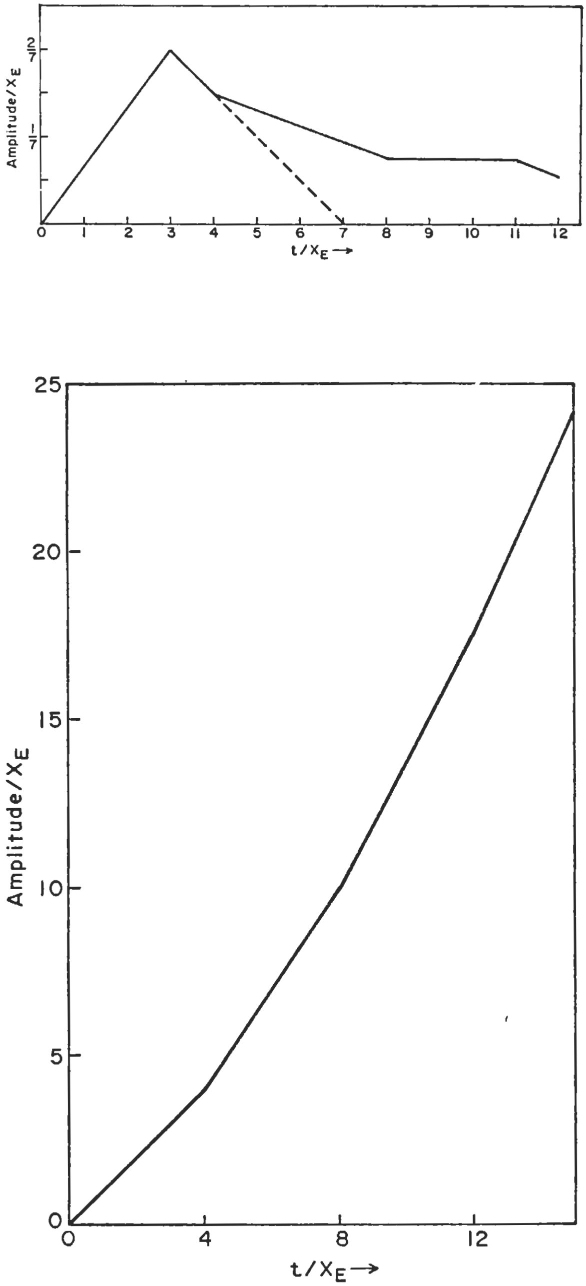


Figure 9. The amplitude of the Kelvin mode at the western boundary $x=0$ in response to a zonal wind stress $F(y)$. (a) The equatorial case $F(y) = \psi_2(y)$. See text for an explanation of the dashed line. (b) The midlatitude case $F(y) = c\psi_J(y)$, $J>>1$. (For the amplitude shown $c = 2^{-3/2}(2J-1)(2J+3)\alpha_0^J$.)

which establishes that the excess at the east is precisely equal to the deficit in the Kelvin mode. The mass flow must be westward to the western coast along the equator in the Kelvin mode, then poleward in the boundary currents and finally eastward in the interior. Figure 6 again serves to illustrate this.

As in case c the flow becomes more complicated at $t=X_E$ when the Kelvin mode reaches the eastern boundary and is reflected as a new series of Rossby modes. An important difference from the earlier case arises because of the linear time dependence of the motions. As the modes propagate across the basin, this translates into a zonal dependence. This zonal dependence extends to all longitudes at $t=3/4X_E$ when the eastward propagating Kelvin mode meets the westward propagating $n=1$ Rossby mode. Since the modes grow until, in some sense, the boundaries are felt, the longitudinal extent of the basin enters as a factor in the amplitude of these modes. In the previous case it influenced only the time scale of the adjustment process. Figure 9a plots the amplitude of the Kelvin mode at the western boundary for $F = \psi_2(y)$. Figure 7a diagrams the arrivals and departures appropriate to this case (except that at $t=0$ only the $n=1$ and $n=3$ modes are excited at $x=X_E$). The unbounded solution consists of only the $n=1$ and $n=3$ modes. The Kelvin amplitude increases linearly until $t=3X_E$ when the $n=1$ Rossby mode arrives from the east, reducing the mass flux into the western boundary and hence the amplitude of the Kelvin mode. If the Kelvin mode somehow never reached the eastern boundary this reduction would continue until $t=7X_E$ when the $n=3$ mode reached the west, bringing the net flux into the boundary to zero and shutting off the Kelvin mode. This is indicated by the dashed line of Figure 9a. Adjustment would then be complete. However, the Kelvin mode does reach the eastern boundary at $t=X_E$ and is reflected as a series of Rossby modes which in turn travel back to the west. Hence the Kelvin amplitude remains positive, as shown.

Figure 9b is a plot of the Kelvin amplitude in response to the forcing $F=c\psi_J(y)$ $J>>1$—a midlatitude zonal wind. Figure 7b diagrams the sequence of reflections. The Kelvin mode shown is due entirely to the initial solution and the reflections of the modes with $n<J$ that arise as components of the eastern boundary reflection of the Kelvin mode at earlier times. The amplitude will continue to increase until $t = (2J-1)X_E$ when the eastern boundary response to the unbounded solution is first felt at the west.

Initial Kelvin mode. In treating $(F)_0\neq 0$ we now need only consider $F=c\psi_0(y)$. The linearity of Eqs. (1) allows the response to a general $F(y)$ to be found by combining this special case with our previous results. When $(F)_0\neq 0$ the unbounded solution contains a secularly growing Kelvin mode. The western boundary response to this Kelvin mode is a free Kelvin mode with the same amplitude and t, x structure as $x-t$. This is not a reflection. The original response is the sum of a locally forced part that goes as x and another eastward propagating part that goes as $t-x$.

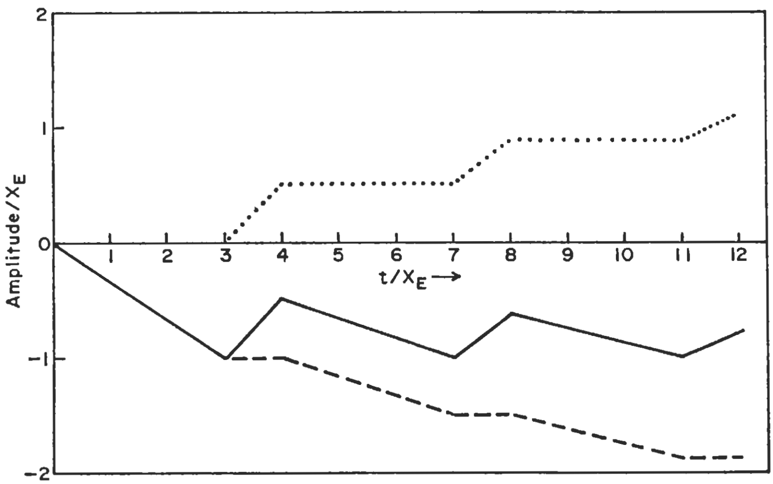


Figure 10. The amplitude of the Kelvin mode at the western boundary in response to $F(y) = 2^{1/2}\psi_0(y)$ (solid line). The dotted line gives the amplitude attributable to the Kelvin mode component of the unbounded response; the dashed line is due to the n=1 Rossby mode component.

The western boundary has the effect of cutting off the forcing to the west of x=0. This results in the propagating part of the original solution being absent for $x<t$, leaving only the locally forced part. The eastern boundary response to Rossby modes discussed in subsection b is qualitatively similar. The Kelvin mode is reflected at the east. Its amplitude there increases linearly until $t=X_E$ when the western boundary effect reaches the east, changing the secular growth to a steady current.

Figure 10 shows the Kelvin amplitude at x=0 in response to $F = 2^{1/2}\psi_0$. In the absence of boundaries the response would be [Eq. (3)*ff*]

$$\mathbf{u}^{(1)} = \underbrace{t\mathbf{M}_{-1}}_{\mathrm{I}} - \underbrace{4/3t\mathbf{R}_1}_{\mathrm{II}} - \underbrace{1/3\mathbf{V}_1}_{\mathrm{III}} \tag{37}$$

The contributions to the amplitude due to term I, the initial Kelvin mode, and that due to term II, the Rossby mode, are shown separately. The former tends to raise the level of water in the basin and the latter to lower it. In this connection note that if a mass source of strength $Q=2^{1/2}\psi_0$ were added to the forcing, term II would be absent in (37), while if a mass sink of the same strength were added term I would be absent. (With Q=0 the mean sea level is unchanged; (36) holds even with a nonzero Kelvin mode).

Returning to just the simple zonal wind forcing the modal decomposition of the steady solution (32) is

$$\mathbf{u} = \mathbf{u}^{(1)} + \underbrace{(x-t)\mathbf{M}_{-1}}_{\text{IV}} + \underbrace{4/3\{[t+3(x-X_E)]\mathbf{R}_1+\mathbf{V}_1\}}_{\text{V}} - \underbrace{X_E\mathbf{M}_{-1}}_{\text{VI}} \qquad (38)$$

with $\mathbf{u}^{(1)}$ given by (37). We can readily trace the creation of all the terms in (38) as the flow evolves. The terms of $\mathbf{u}^{(1)}$ are the unbounded response; IV results from I at the western boundary and V from II at the eastern boundary. In both cases the boundary acts as a cut-off for the forcing rather than reflecting an incident motion. Term VI is the reflection at the west of the sum of the Rossby modes II and V; the Kelvin mode I+IV+VI has zero amplitude at the east so no reflection is required there. Eq. (38) is not established everywhere in the basin until $t=4X_E$. Initially, mode II is reflected at the west as a Kelvin mode of amplitude $1/3(x-t)$; at $t=3X_E$ mode V arrives at $x=0$ changing this amplitude to give mode VI; mode VI does not cross the basin until $t=4X_E$. Until that time the Kelvin mode amplitude was nonzero at $x=X_E$ resulting in reflected Rossby modes. (These modes carry a mass excess of $2\pi^{1/4}X_E^2$; (38) alone does not satisfy the mass conservation condition).

In summary, all the modes comprising (38), [or, equivalently, (32)] are set up by $t=4X_E$, but this steady response is not the total response. Additional motions are initiated in the time before (38) sets up and these motions continue travelling back and forth across the basin for all time. Eq. (38) implies that the Kelvin amplitude at $x=0$ is $-X_E$; Figure 10 shows that the oscillating motions give a Kelvin amplitude that is a substantial fraction of that. The evolution of this flow may be likened to the result of tilting a nonrotating pan of water: the steady state is level but in the absence of friction there would be endless (inertia-gravity wave) motions travelling back and forth across the basin.

It should be clear that the failure to reach a steady state is not due to the failure of (38) to satisfy the mass conservation constraint. For $F=2\psi_2-2^{1/2}\psi_0$ this constraint is satisfied and all the modes needed to synthesize (38) are set up by $t=8X_E$. Nevertheless, other motions, initiated before this time, give rise to persistent reflections. Figure 11 illustrates this: the amplitude of the oscillations decreases very slowly. Furthermore, the failure to reach a steady state is not simply a consequence of taking the basin to be meridionally infinite, since the low n modes would be little altered by zonal boundaries. (This is true as long as these are in midlatitudes and not close to the equator; see the discussion in I, Sec. 4). A steady state is not reached because the planetary equatorial modes travel at different speeds, and in different directions (i.e. the Kelvin mode). In addition, they reflect as a series of modes rather than a single mode. For all of these reasons the mass fluxes cannot be synchronized to give only a steady response.

e. $Q\neq 0$. We will not discuss in detail the response to a mass source Q. Most of its characteristics are similar to the response to a zonal wind forcing. If $\overline{Q} \equiv \int\int Q$

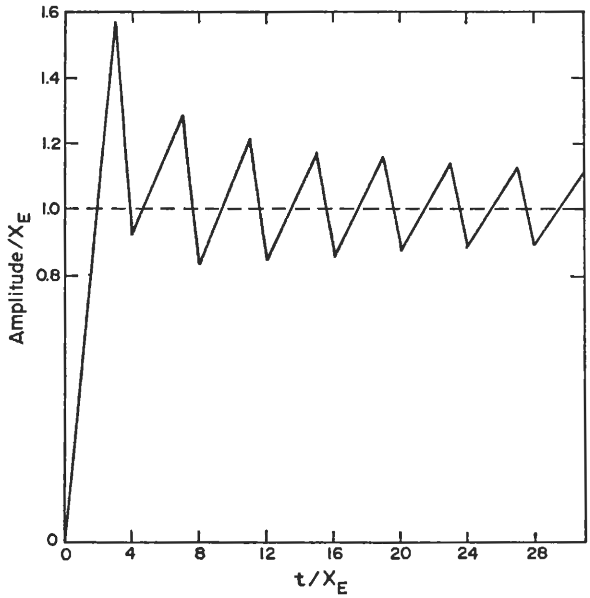


Figure 11. The amplitude of the Kelvin mode at the western boundary in response to $F(y) = 2\psi_2(y) - 2^{1/2}\psi_0(y)$. The dashed line is the steady state value.

$= 0$ then a steady solution of the form (25) is possible. If $Q(y)$ is antisymmetric about the equator the time dependent response goes to this steady state at the eastern boundary in the same way that it did for F antisymmetric, case b above. Q symmetric (but still with $\overline{Q}=0$) gives a response qualitatively like case d, F symmetric. The flow evolves the components of the steady solution but generates additional motions along the way.

If $\overline{Q}\neq 0$ then a steady response is not possible: if $\overline{Q}$ is positive (negative) there is a net source (sink) of mass and $\int\int h$ must increase (decrease) with time. Rewrite an arbitrary $Q(y)$ as $\{Q(y)-Q^*\}+Q^*$ with $Q^*=\overline{Q}/2Y^*$ for $|y|<Y^*$, and $Q^*=0$ otherwise. Then the bracketed term is neither a net mass source or sink and the discussion in the previous paragraph implies. Note that for a uniform mass source, say $Q=c$ everywhere, the response is simply $u=v=0$ and $h=ct$. The response to Q^* will be approximately this for $|y|<Y^*$ if $Y^*>>1$. Putting these results together the response to a $Q(y)$ with $\int Q(y)dy>0$ is as described in the previous paragraph with the addition of a constant rise in sea level everywhere.

6. Conclusion

We have obtained the solution for the linear spin-up of a meridionally bounded equatorial ocean in response to a switched-on, x-independent wind stress in terms of only five basic components:

(i) The unbounded nonoscillatory response $\mathbf{u}^{(4)}$ to the wind stress forcing [Eqs. (4)-(6)];
(ii) the inertia-gravity waves generated when the forcing switches on, together with their reflections;
(iii) the eastern boundary response, consisting of packets of long-wave Rossby waves, needed to bring $u^{(4)}$ to zero and reflect subsequent Kelvin waves emitted by the western boundary;
(iv) the ever thinning western boundary layer made up of boundary trapped short-wave Rossby packets that, together with the Kelvin waves, is needed to bring $u^{(4)}$ to zero and reflect any subsequent Rossby waves emitted at the eastern boundary;
(v) the equatorial Kelvin wave component of the western boundary response, needed to return eastward all of the net (i.e., meridionally integrated) mass flux incident on the western boundary.

We had previously seen (in I) that the inertia-gravity waves play the same role for an unbounded equatorial ocean that they do for a midlatitude or f-plane ocean: they effect the movement of mass that is necessary to establish the low frequency unbounded circulation. In the present work we have seen that they play no role in adapting this circulation to the presence of boundaries. We have further seen that a boundary influences the response in two distinct ways: the first is as a barrier, reflecting motions that propagate energy toward it; the second is as a cutoff of the forcing, eliminating the possibility of motions propagating into the basin.

In tracing the evolution of the low frequency flow the basic question we have tried to answer is whether or not the time-independent solutions of the inviscid shallow water equations are reached: if so, how so; if not, why not. We have seen that the answer to the question depends on whether or not a Kelvin wave is allowed by the symmetry of the forcing, for there is a net (i.e., meridionally integrated) zonal transfer of mass only with a Kelvin wave.

For those symmetries for which a Kelvin wave is forbidden ($G(y)$ symmetric, $F(y)$ antisymmetric), spin-up occurs entirely by the effects of the Rossby waves emanating from the eastern boundary. As more and more Rossby waves reach an interior point, the state gets closer and closer to the steady solution. Thus points closer to the eastern boundary spin up first, the entire process occurring more slowly as we recede meridionally from the equator. Meridional transports of mass are accomplished largely by the trapped boundary layer motions on the western boundary.

From those symmetries for which a Kelvin wave is permitted ($G(y)$ antisymmetric, $F(y)$ symmetric), the Rossby waves emitted by the eastern boundary act to bring the sea surface *tilt* to the steady value and the spin-up of the sea surface tilt proceeds as in the non-Kelvin cases. A minor additional feature appears if the unbounded response has a Kelvin mode component. In this case a part of the tilt (the

Kelvin part) is set up from west to east, starting at the western boundary. A more substantial difference is that a final steady state is not reached as the tilt is set up. In the course of the spin-up process described thus far additional motions are excited that in effect slosh a certain amount of mass back and forth across the basin: eastward in the form of Kelvin waves and back again westward in Rossby waves. It is not until the Kelvin wave slowly loses amplitude through successive reflections at the eastern and western boundaries that the sea surface height approaches its final steady value. This latter process proceeds very slowly, depending as it does on successive traversal of the Kelvin and Rossby waves across the basin.

The distinctive properties of the spin-up are due to the specific nature of the boundary response, namely that the low frequency eastern boundary response is an infinite series of westward propagating Rossby waves less equatorially confined than the incident motions, and that, in addition to a boundary trapped part, the low frequency western boundary response may contain a rapidly propagating equatorially confined Kelvin wave. The different motions travel at different speeds and in different directions and so cannot become synchronized. The result is that these motions cannot readily spread the mass associated with them evenly over the basin as required for a steady state equilibrium. The motions overshoot and undershoot the final state giving rise to long period, slowly diminishing, oscillations about that state. By contrast, for motions governed by the batrotropic vorticity equation the eastern boundary response is a single Rossby mode at the meridional wave number of the incident motion; spin-up for that wave number is complete when the boundary Rossby mode crosses the basin to the western side.

The fundamental time scale of the spin-up process, T_A, is the time it takes for an equatorial Kelvin wave to cross the ocean basin. For values typical of the first baroclinic mode (cf. I, Sec. 1 or Moore and Philander, 1976) T_A is three months, 37 days, and 34 days for the Pacific, Atlantic, and Indian Oceans, respectively. For symmetries that exclude the Kelvin wave the successive Rossby modes arrive at the west at times $5T_A$, $9T_A$, $13T_A$, . . . with each successive arrival increasing the meridional extent of the adjusted region. When the Kelvin wave is present the boundaries are first "felt" at all longitudes at the equator at time $3/4\ T_A$ when the Kelvin and $n=1$ Rossby waves meet. The oscillations that occur with the Kelvin wave symmetry have a period of $4T_A$—the time T_A for the Kelvin wave to cross the basin plus the times $(4m+3)T_A$, $m=0,1,2,$ . . . for the Rossby waves with odd index to cross the basin. For the Pacific this period is one year, suggesting the possibility that these oscillations enhance the response to the annual wind stress cycle. In any case, even at the equator it takes longer than one year for the circulation to adjust to changes in the wind stress. (This is true for all three oceans).

Our results show that the wave guide effect of the equator makes equatorial currents predominately zonal, with the exception of those near the western boundary. This is, of course, consistent with the observed circulations in the equatorial oceans (including the Indian Ocean; see Sharma, 1976). The analysis of I shows this result will hold for wind stress with zonal variations with the exception of meridional winds stresses with (sharp) zonal gradients (for example, the Somali jet during the south-west monsoon). We have seen that the restriction of strong meridional currents to the western boundary holds even when spin-up requires substantial cross-equatorial redistributions of mass at all longitudes.

It is important to emphasize that the model equations we have solved are of limited applicability to the real world because of the certain importance of nonlinear and viscous effects (cf. I, Sec. 7). Nevertheless, in addition to their direct applications the solutions we have obtained are useful conceptual tools for the study of more realistic numerical models (e.g., Cane, 1975, 1976; Hurlburt *et al.*, 1976).

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APPENDIX

The summation formulas that follow are used in the body of this paper. In all cases J is an odd integer.

$$\textit{Lemma:}\quad S_J \equiv (\alpha_1^J)^2\,(J+1)^{-1} + \sum_{n=1}^{J} (\alpha_1^n)^2\,[n(n+1)]^{-1} = 1 \tag{A1}$$

Proof: If $J=1$, $S_J = ½ + ½ = 1$. Assume (A1) is true for $j=J$. Now

$$S_{J+2}-S_J = (\alpha_1^{J+2})^2\,(J+3)^{-1} + (\alpha_1^{J+2})^2\,[(J+2)\,(J+3)]^{-1} - (\alpha_1^J)^2\,(J+1)^{-1}$$
$$= (\alpha_1^J)^2\,\{(J+2)\,[(J+1)\,(J+3)]^{-1} + [(J+1)\,(J+3)]^{-1} - (J+1)^{-1}\} = 0.$$

Hence if $S_J = 1$ then $S_{J+2} = 1$ and by induction (A1) holds for all J.

$$\textit{Lemma:}\quad \sum_{n=1}^{\infty} (\alpha_1^n)^2\,[n(n+1)]^{-1} = 1. \tag{A2}$$

$$\text{Proof: } 1 = \lim_{J\to\infty} S_J = \lim_{J\to\infty} \left\{ \sum_{n=1}^{J} (\alpha_1^n)^2\,[n(n+1)]^{-1} + O\,(J^{-1/2}) \right\}$$

From (A1) and (A2)

$$\sum_{n>J}^{\infty} (\alpha_J^n)^2\,[n(n+1)]^{-1} = [J+1]^{-1} \tag{A3}$$

and

$$[(J+1)\,(2J+1)]^{-1} + \sum_{n>J}^{\infty} (\alpha_J^n)^2\,[n(n+1)]^{-1} = 2\,[2J+1]^{-1} \tag{A4}$$

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