A Mathematical Note on Kawase's Study of the Deep-Ocean Circulation*

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1. Introduction

Kawase (1987) recently presented an enlightening extension of the celebrated but underutilized Stommel-Arons model of the deep circulation. The principal change from that by now classical formulation is the replacement of a prescribed upwelling at mid-depth with one parameterized in terms of the layer thickness determined by the flow. For weak thermal damping Kawase's results are similar to Stommel-Arons. However, he explicitly connects the equatorial region to higher latitudes, and his principal finding is that for strong thermal damping the deep western boundary current emanating from a high latitude source separates, turns eastward at the equator, and finally turns poleward into both hemispheres at the eastern boundary.

The purpose of the present note is to offer a succinct and nearly exact solution as a replacement for the approximate and elaborate mathematical development in Kawase's paper. It is hoped that this will make the original more accessible and more amendable to additional applications. The physical formulation and justification, along with a cogent interpretation of the results may be found in the original and will not be reprised here.

2. Mathematical development

We begin with the steady, linear shallow water equations on an equatorial beta plane, Kawase's Eqs. (3.1)–(3.3):

$$Ku^* - \beta y^* v^* = -g\frac{\partial h^*}{\partial x^*}, \qquad (1)$$

$$Kv^* + \beta y^* u^* = -g\frac{\partial h^*}{\partial y^*}, \qquad (2)$$

$$\lambda h^* + H\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}\right) = 0. \qquad (3)$$

Here H is the equivalent depth for this one mode ocean, K a Rayleigh friction coefficient and λ, the Newtonian damping, is the consequence of parameterizing mid-depth upwelling to balance diffusion. The basin is bounded by meridional walls at $x^* = 0$ and $x^* = L_B$. Instead of an explicit deep-water formation region, the flow is driven by a specified inflow $U_0^*(y)$ on the western boundary at high northern latitudes. Elsewhere on the boundaries $u^* = 0$.

The asterisks mark dimensional variables; in the canonical equatorial scaling one defines wave speed, length and time scales by the relations:

$$c_0 = (gH)^{1/2}; \quad L_0 = (c_0/\beta)^{1/2}; \quad T_0 = (c_0\beta)^{-1/2}. \qquad (4)$$

Instead, here we will first multiply (3) by K/λ so the Rayleigh damping coefficients in all three equations appear to have the same value, K, and the equivalent depth seems to be $(K/\lambda)H$. Now define the analogous scales:

$$c = \left[\frac{K}{\lambda}gH\right]^{1/2}; \quad L = (c/\beta)^{1/2}; \quad T = (c\beta)^{-1/2}. \qquad (5a)$$

Hence

$$c = \left(\frac{K}{\lambda}\right)^{1/2} c_0; \quad L = \left(\frac{K}{\lambda}\right)^{1/4} L_0; \quad T = \left(\frac{K}{\lambda}\right)^{-1/4} T_0. \qquad (5b)$$

As usual, relate scales for currents U and height D by $U = gD/c$. Applying the scaling (5) transforms (1)–(3) to the standard nondimensional form with nondimensional Rayleigh coefficient $K' = KT = (K^3\lambda)^{1/4}T_0$. If one also wrote $K' = i\omega$ the set of equations would have the form usually used to derive equatorial waves. By drawing on the extensive body of literature pertaining to these equations the problem at hand may be solved relatively painlessly.

We follow Kawase in taking the momentum dissipation K to be small, though with a technical change: Kawase assumes $\epsilon \equiv K/\beta L_B \ll 1$ while here the more restrictive, but still realistic assumption is made that $K' \ll 1$. The low-frequency approximation then applies,

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in which case all of the mass flux incident on the western boundary is returned in an equatorial Kelvin wave (Cane and Sarachik 1977). The role of the western boundary current is to provide the equatorward transport which makes this possible. The incident flux consists of the specified input U_B together with any additional generated as part of the solution.

The Kelvin wave leaves the western boundary and crosses to the east, losing amplitude to dissipation along the way. Let us say it arrives at $x^* = L_B$ with amplitude A. Cane and Moore (1981) show that the sum of this wave and its reflection in Rossby waves has the closed form

$$[u, v, h] = A(\cosh 2r\zeta)^{-1/2} \exp\left(\frac{-y^2}{2} \tanh 2r\zeta\right) \times [\sinh 2r\zeta, K'y \operatorname{sech} 2r\zeta, \cosh 2r\zeta] \quad (6)$$

where $\zeta = (L_B - x^*)/L_B$ ranges from 1 to 0 as the basin is traversed from west to east. The damping parameter $r = KL_B/c = K'L_B/L$ is related to Kawase's ϵ by $r = (K/\lambda)^{1/2}\epsilon$. [Equation (6) is exactly the Cane and Moore mode with K' substituted for $i\omega$; the interested reader is referred to the derivation in that paper, which is neither long nor arduous.]

The total solution to Kawase's problem consists of three parts: (i) The inflow $U^B(y)$ at $\zeta = 1$; (ii) the interior flow (6); (iii) the western boundary current. The as yet unknown constant A appearing in (6) is determined by the previously mentioned condition (Cane and Sarachik 1977) that exclusive of the boundary current (iii), the net zonal mass flux at the western boundary must be zero. Using the form of u given by (6) at $\zeta = 1$:

$$0 = -\int_{-\infty}^{+\infty} U^B(y)dy + A(\cosh 2r)^{-1/2} \sinh(2r) \times \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2} \tanh 2r\right)dy. \quad (7)$$

Letting $s = y[\tfrac{1}{2} \tanh 2r]^{1/2}$,

$$0 = -\int_{-\infty}^{+\infty} U^B(y)dy + A[\sinh 2r]^{1/2} \sqrt{2} \int_{-\infty}^{+\infty} e^{-s^2} ds;$$

$$A = [2\pi \sinh 2r]^{-1/2} \int_{-\infty}^{+\infty} U^B(y)dy. \quad (8)$$

Once A is known, the western boundary current is determined as well (see Cane and Sarachik 1977, p. 404). With u given by (6) its transport V is determined by

$$V = \int_{y}^{+\infty} [-U^B(y') + u(\zeta = 1, y')]dy'. \quad (9)$$

It follows immediately from (9) that $V = 0$ at $y = \infty$ while (7) insures that $V = 0$ at $y = -\infty$. For this Rayleigh friction model the boundary layer structure is that of the Stommel model.

This completes the solution, but two technical points require comment. The problem has been solved for a meridionally infinite ocean, where as Kawase (and nature) impose boundaries at $|y| = Y$, say. If Y is far from the equator the solution may be amended by adding a westward flowing boundary layer at Y which accepts all the poleward mass flux implied by (6) at that latitude. This well known matching condition was proved rigorously by Moore (1968) and is discussed in many places (e.g., Cane and Patton 1984). The only change it makes is that boundary current (essentially a coastal Kelvin wave) is much swifter than the high latitude Rossby waves it replaces so less mass flux is dissipated on the journey eastward than implied by (6). This affects the matching condition (7), but the percentage of the total flux affected, which goes like Y^{-2}, is small for large Y.

Second, (6) assumes the long-wave, low frequency approximation [which amounts to neglecting Kv^* in (2)]. For large enough damping K this does not strictly hold. However, in this case the flow is trapped to the equator and eastern boundary, and although (6) is incorrect in detail, it still captures the large scale features of interest here (see Figs. 4–7 of Cane and Moore 1981).

The solution (6), (8) is structurally different for different parameter values, as discussed by Kawase. The parameter exerting primary control is $r = (K\lambda)^{1/2}L_B/c_0$. Here L_B/c_0 is the time it takes for a Kelvin wave to cross the basin, and $(K\lambda)^{-1/2}$ is the harmonic mean of thermal and momentum damping times. Thus r is a measure of how strongly a wave is damped in crossing the basin.

If the damping is strong enough so $\tanh r\zeta \approx 1$ (i.e. r and $r\zeta \gtrsim 1$) then (6) is approximately

$$[u, v, h] \approx \pi^{-1/2}\left[\int U^B dy\right] \frac{\sqrt{2}}{4} e^{-y^2/2} e^{-r(1-\zeta)} \times [1, 2K'ye^{-4r\zeta}, 1]. \quad (10)$$

The meridional velocity v is small everywhere, and u and h fall off rapidly away from the equator. Dimensionally the e-folding scale,

$$y^* \sim \sqrt{2}\left[\frac{\lambda\beta^2}{KgH}\right]^{1/4} \quad (11)$$

depends on the ratio of the damping times as well as the equivalent depth. Note that u and h also weaken away from the western boundary with dimensional decay scale $r^{-1}L_B$.

Even with $r \gtrsim 1$ (10) does not apply near the eastern boundary where $r\zeta \ll 1$. In this region

$$[u, v, h] \approx \pi^{-1/2}\left[\int U^B dy\right] \frac{e^{-r}}{2} e^{-y^2 r\zeta}[2r\zeta, K'y, 1]; \quad (12)$$

zonal velocity is small and the meridional velocity is strong and poleward, but only within a boundary layer of thickness $\zeta \sim (ry^2)^{-1}$. Dimensionally

$$L_B - x^* \sim \frac{gH}{\lambda\beta}(y^*)^{-2}, \tag{13}$$

independent of the momentum damping K. In sum, for strong damping the flow is a western boundary current flowing from the source region to the equator. There it turns eastward, weakening as it goes. At the eastern boundary it divides into a poleward flow in both hemispheres which narrows as the latitude increases. The circulation is neglible outside of these boundary layers. This is the flow depicted in Fig. 3 of Kawase.

For weak damping, $r \ll 1$, $\sinh 2r \approx 2r$, $\tanh 2r\zeta \approx 2r\zeta$, $\cosh 2r\zeta \approx 1$, so (6) is approximately

$$[u, v, h] \approx \pi^{-1/2} \int U^B dy \frac{r^{-1/2}}{2} e^{-y^2 r\zeta} [2r\zeta, K'y, 1]. \tag{14}$$

Define $\eta = y^*/L_B$ so that, as in Kawase, latitude and longitude are both scaled by the basin width. Now rewrite (14) as

$$[u, v, h] \approx \pi^{-1/2}\left[\int U^B dn\right] \delta^{-1/2} e^{-\delta\eta^2\zeta} [2\zeta, \eta, 1] \tag{15}$$

where

$$\delta = \frac{\lambda\beta L_B{}^3}{gH}$$

is the parameter at the core of Kawase's analysis. The angle of the flow from east is $\tan^{-1}(\eta/2\zeta)$: the flow is predominantly zonal at low latitudes near the west, becoming more meridional as both latitude and longitude increase. The interior flow remains strong out to a latitude $y^* \sim L_B\delta^{-1/2}$; see Kawase, Figs. 4 and 5.

The condition (7) sets the Kelvin wave amplitude at the western boundary to return all the mass flux incident there. If the damping is high this is just the imposed mass flux U_B [cf. (10) for $\zeta = 1$]. When the damping is low, as in (14), then it also includes Rossby waves generated at the eastern boundary as the reflection of the Kelvin wave. In the former case these are damped out before they cross the basin, whereas with weak damping the flow must fill the interior in order to achieve enough dissipation to balance the inflow. It is interesting that the solution is strongly controlled by the thermal damping λ [viz. (13), (15)], but is almost indifferent to the momentum damping K [but cf. (11)].

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REFERENCES

Cane, M. A., and E. S. Sarachik, 1977: Forced baroclinic ocean motion II: The equatorial unbounded case. *J. Mar. Res.*, **35**, 395–432.

——, and D. W. Moore, 1981: A note on low frequency equatorial basin modes. *J. Phys. Oceanogr.*, **11**, 1578–1584.

——, and R. Patton, 1984: A numerical model for low frequency equatorial dynamics. *J. Phys. Oceanogr.*, **14**, 1853–1863.

Kawase, M., 1987: Establishment of deep ocean circulation driven by deep-water production. *J. Phys. Oceanogr.*, **17**, 2294–2317.

Moore, D. W., 1968: Planetary-gravity waves in an equatorial ocean. Ph.D. thesis, Harvard University.