FINITE DIFFERENCE METHODS FOR ACOUSTIC AND ACOUSTO-GRAVITY WAVEFIELDS: APPLICATION TO LOW FREQUENCY INFRASOUND PROPAGATION

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ABSTRACT

A finite difference (FD) method is derived for the frequency-domain acoustic and acousto-gravity equations as expressed in cylindrical coordinates. First, it is shown analytically that for an isovelocity atmosphere in which the density decreases exponentially with altitude, the acoustic wave equations (i.e., neglecting gravity) indicate that (1) the pressure varies with the square root of the density and (2) the acoustic wavelength depends not only on the frequency and velocity but also on the rate of exponential decrease of the density with altitude. Discretization of the linear equations relating acoustic particle velocities and pressures is straightforward except near the source. At r = 0, a singularity exists for the acoustic wave equation as expressed in cylindrical coordinates. In this case, l'Hopital's rule is used to transform the singularity at r = 0 into a determinate form. The FD equations must be solved simultaneously, which requires the solution of a very large, sparse matrix equation, which is accomplished using a quasi-minimum residual method. The resulting FD algorithms can be used to solve for sound intensities in arbitrarily complex models that may include high material contrasts and arbitrary topography.

The FD method is also extended to handle the effects of gravity on the acoustic wavefields. It is shown that the acousto-gravity equations require the simultaneous solution of the pressure and density perturbations caused by the passage of a sound wave. Acousto-gravity waves are relevant at very low frequencies, especially in the upper atmosphere where densities are very low.

OBJECTIVES

A FD method of solving the acoustic wave equation in cylindrical coordinates has been developed. The FD method yields the solution to a discretized version of the full acoustic wave equation for arbitrarily complex media. It is a full spectrum approach and is thus reliable at all angles of propagation, including backscatter. This offers an advantage over other standard propagation methods in wide use, as it allows for accurate computation of acoustic energy levels in the case where significant scattering can occur near the source, such as may happen for an explosion near the surface, or underground. This fits in with nuclear monitoring goals in that it allows for an improved understanding of the generation and propagation of infrasound energy from underground and near-surface explosions.

At high frequencies, the atmospheric density varies insignificantly over the scale of the acoustic wavelength and hence computation of the acoustic propagation is quite straightforward. However, atmospheric density drops off by an order of magnitude with approximately every 15–20 km in altitude. As will be shown in this paper, these variations in density strongly affect acoustic propagation at very long periods—over 100 s—for which the acoustic wavelength is greater than 30 km. FD solutions are presented for very low frequency sources for the acoustic wave equation. An FD computational method is sketched out for acousto-gravity propagation.

RESEARCH ACCOMPLISHED

Equations Governing Acoustic and Acousto-Gravity Propagation

The standard equations governing sound propagation may be written in the frequency domain as

$$-i\omega p = -\rho_0 c^2 \nabla \cdot \mathbf{v} \qquad (1a)$$

$$-i\omega \mathbf{v} = -\frac{1}{\rho_0} \nabla p \qquad (1b)$$

where the convention $p = p\, exp(-i\omega t)$ has been used to transform from the time to the frequency domain. In the above equations, p denotes the acoustic pressure, $\mathbf{v}$ is the acoustic particle velocity vector, ρ_0 is the ambient density in the absence of perturbations caused by the sound wave, c is the propagation speed of the medium, and ω is the circular frequency.

Buoyancy effects are significant in the propagation of acoustic energy at very low frequencies. The following acousto-gravity equations combining buoyancy and compressibility effects govern sound propagation at very low frequencies:

$$-i\omega p = \mathbf{v} \cdot (\rho_o \mathbf{g}) - \rho_o c^2 \nabla \cdot \mathbf{v} \qquad (2a)$$

$$-i\omega \mathbf{v} = -\frac{1}{\rho_o} \Big[\nabla p + \rho_s \mathbf{g} \Big] \qquad (2b)$$

$$-i\omega \rho_s = -\mathbf{v} \cdot \nabla \rho_o - \rho_o \nabla \cdot \mathbf{v} \qquad (2c)$$

where $\mathbf{g} = (0,0,9.8)$ is the acceleration that is due to gravity and ρ_s is the density perturbation caused by the passage of the sound wave. The term $\rho_s \mathbf{g}$ in Eq. (2b) is a buoyancy force (Gill, 1982); note that a negative density perturbation yields an upward acceleration that is due to the effect of gravity. Note that, in the case where gravity contributes negligibly, Eqs. (2) reduce to Eqs. (1). In that case, the density perturbations are approximately a scalar multiple of the acoustic pressure, i.e., $\rho_s = p/c^2$, since the second term in Eq. (2c) dominates.

The standard FD method relies on replacing linear partial differential equations with a set of discrete equivalents. Field solutions are then computed over a discrete set of nodes that compose the spatial grid. Figure 1 indicates how the field variables are defined and how the medium is discretized for the solution method presented in this paper.

The FD method is formulated in cylindrical coordinates here. A radially symmetric model is assumed, allowing for the response to a continuous wave point source to be computed using only two coordinate directions.

The model is decomposed into a set of discrete cells of dimension $\Delta r \times \Delta z$, each with a uniform propagation speed c and ambient density ρ_0. Pressure nodes are defined at the center of each cell, and the velocity variables are located midway between the pressure nodes. The staggered grid formulation increases the accuracy of the FD solution, since central differences are used to compute the discrete derivatives (Taflove and Hagness, 2000). A column of cells of width $\Delta r/2$ bounds the model at the axis of symmetry at $r = 0$; these become full cells when reflected about the axis. This choice allows pressure nodes, and hence the source, to be defined along the axis of symmetry. The locations of the vertical velocity nodes are defined in such a way as to allow a rigid surface ($v_z = 0$) to be defined at the bottom of the model.

Variables for the density perturbations must also be introduced when buoyancy effects are included in the FD solution. In the first-order equations above, both the density and pressure perturbations are dependent on each other only through the equation for the velocity perturbations. Therefore, the density variables ρ_s may be located at the same points in space and time as the pressure variables p.

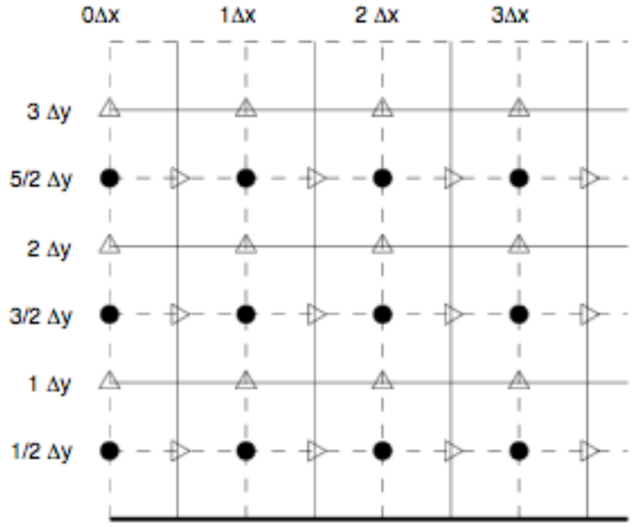

Figure 1. The FD model is decomposed into a set of discrete cells, indicated by the solid lines, each with uniform velocity *c* and ambient density ρ_0. The acoustic pressure and density perturbations are defined by the nodes at the center of each cell, indicated by the filled circles. When gravitational effects are neglected, only pressure perturbations need to be computed. The locations of the horizontal velocity v_r (triangles pointing right) and vertical velocity variables v_z (triangles pointing up) are defined on a spatially staggered grid, as shown.

Analytic Solution for an Isovelocity Model with Exponentially Decreasing Density

In a realistic atmospheric model, the ambient density decreases exponentially with altitude. An example of a realistic velocity and density profile is shown in Figure 2. In this section an analytic solution is derived for an isovelocity atmospheric model in which density decreases exponentially with altitude.

It can be shown that for a model with uniform velocity and exponentially decreasing density, where $\rho_0(z)=\rho_0(0)\, e^{-az}$, Eqs. (1) can be combined to yield

$$k^2 p + a\frac{\partial p}{\partial z} + \nabla^2 p = 0, \tag{3}$$

where k = ω/c is the wave number. Using a change of variables $p(r,z) = e^{-az/2}\, q(r,z)$, where the exponential term is equal to the square root of the density, Eq. (3) becomes

$$(k^2 - a^2/4)q + \nabla^2 q = 0, \tag{4}$$

after some manipulation. The *q* field values depend only on distance from the source (variability in depth was removed by the change of variables). By solving Eq. (4) in spherical coordinates (Jensen et.al., 1994) and converting back to cylindrical coordinates, it can be shown that

$$q(r,z) = A_1\frac{e^{ik_m R}}{R} + A_2\frac{e^{-ik_m R}}{R}; \tag{5}$$

where

$$k_m = \sqrt{k^2 - a^2/4}; \quad R = \sqrt{r^2 + (z - z_s)^2}$$

are the modified wave numbers and distance from source, respectively, and z_s is the source altitude. The pressure for an isovelocity whole-space with exponentially decreasing density, bounded by a rigid halfspace is given by

$$p(r,z) = \sqrt{\rho_0}\left[A_1\frac{e^{ik_m R}}{R} + A_2\frac{e^{-ik_m R}}{R}\right]; \tag{6}$$

where the first term corresponds to diverging spherical waves and the second term to converging spherical waves. For a wholespace, $A_2 = 0$. Strictly, this is not a physically realistic model since the density is defined as exponentially decreasing with z, or equivalently, exponentially increasing as z approaches negative infinity. The pressure thus also approaches infinity in this direction. A realistic analytic model would have to include a lower boundary with a realistic density; such a model is examined in the section on the FD method.

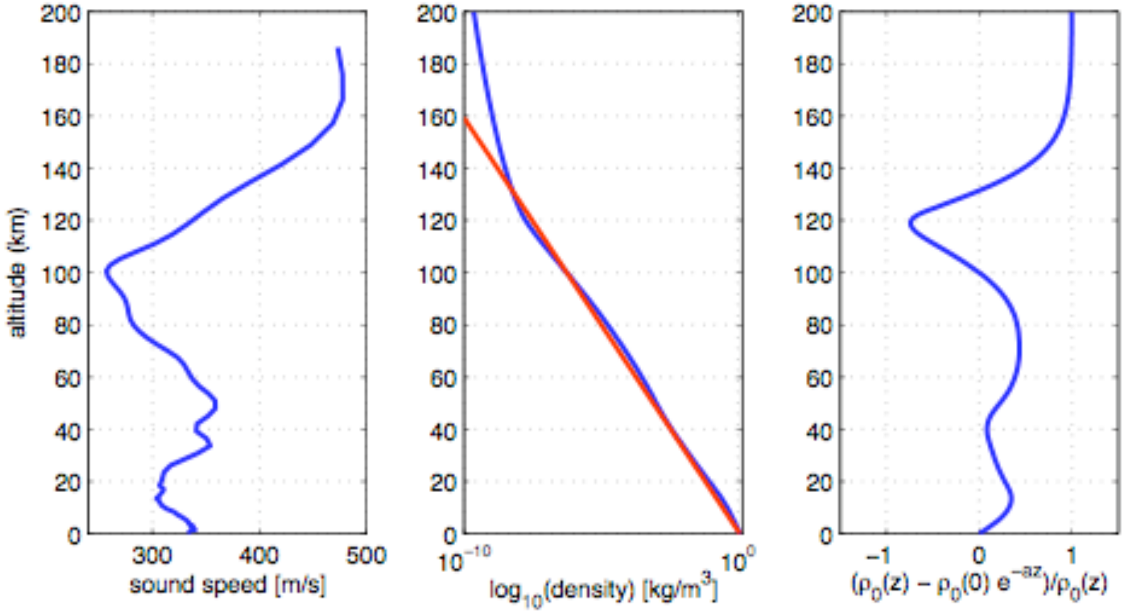

Figure 2. Profiles for atmospheric sound speed (left) and density (blue line, center). The red line at center shows the density profile corresponding to an exponentially decreasing density profile of e^{-az} profile, where a = 1.457×10^{-4}. The profile at right shows the fractional difference between the actual and exponential profiles. As indicated, the exponential approximation is adequate to an altitude of nearly 150 km.

Note that the pressure solution indicates that the acoustic wavelength, which is inversely proportional to the wave number, depends not only on the frequency and sound speed but also on the exponential decrease of density with altitude, which is characterized by the value of a, as defined in Eq. 3. Figure 3 shows a plot of wavelength λ vs. period, for a medium with a uniform velocity of c = 300 m/s. The relation is linear until k is on the same order of magnitude as a. For very, very low frequencies, Eq. (6) suggests that the acoustic wavefield should undergo exponential decay as k_m becomes imaginary. However, note that this derivation neglects the effects of gravity; these effects are discussed in the section on the FD solution to the acousto-gravity problem.

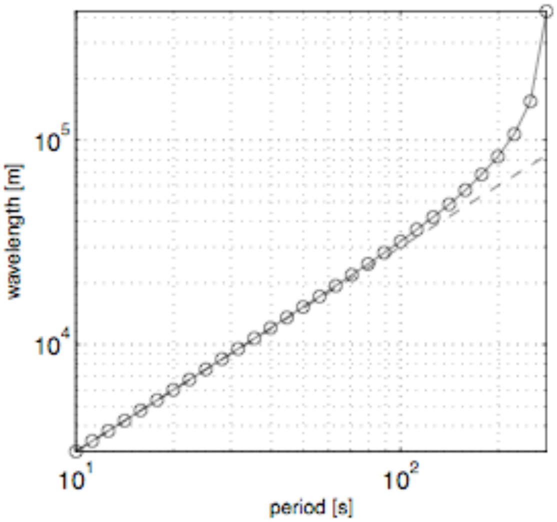

Figure 3. **Wavelength k_m vs. period for an isovelocity medium (c = 300 m/s), with exponentially decreasing density. The solid line indicates the acoustic wavelength for an atmosphere with exponentially decaying density, where the decay rate a is set to 1.457 x 10^{-4}, i.e., the same value as for the realistic density profile shown in Figure 2. The dotted line indicates the wavelength for $a = 0$. In the latter case, the period and wavelength vary linearly. At even longer periods, the k_m becomes an imaginary value.**

FD Solution for Acoustic Propagation in the Atmosphere

In cylindrical coordinates, the FD solution of the frequency-domain acoustic wave equations involves solving the discretized equivalents of Eq (1), i.e.,

$$0 = i\omega p_{i,j} - \rho_{o_{i,j}} c_{i,j}^2 \left[\frac{v_{r_{i,j}} - v_{r_{i-1,j}}}{\Delta r} + \frac{v_{r_{i,j}} + v_{r_{i-1,j}}}{r} + \frac{v_{z_{i,j}} - v_{z_{i,j-1}}}{\Delta r} \right] \qquad (7a)$$

$$0 = i\omega v_{r_{i,j}} - \frac{2}{\rho_{o_{i+1,j}} + \rho_{o_{i,j}}} \left[\frac{p_{i+1,j} - p_{i,j}}{\Delta r} \right] \qquad (7b)$$

$$0 = i\omega v_{z_{i,j}} - \frac{2}{\rho_{o_{i,j+1}} + \rho_{o_{i,j}}} \left[\frac{p_{i,j+1} - p_{i,j}}{\Delta z} \right] \qquad (7c)$$

where i indicates the row index and j indicates the column index, and v_r and v_z are the radial and vertical components of the velocity, respectively.

All equations must be solved simultaneously, which implies that a matrix equation of the type

$$\mathbf{A}\,\mathbf{x} = \mathbf{b} \qquad (8)$$

must be solved, where **x** is the $N \ x \ 1$ vector of field components sought (in this case, the pressure variables at each node); **b** is the $N x 1$ source vector, mainly composed of zeros; and **A** is a large, sparse matrix of size $N x N$. The problem is made feasible by the fact that only approximately $5 \ x \ N$ variables of **A** are non-zero and need to be stored. For large models, the total number of variables can be on the order of tens to hundreds of thousands; therefore, a highly efficient method is required for solving this problem. The quasi-minimal residual (QMR) algorithm, developed to handle large, linear systems that have only a few non-zero entries per row (Freund et.al., 1991), was used to solve the examples presented in this paper.

In Figure 4, solutions of these equations are shown for 3 simple cases. In each case the bottom boundary is a flat, rigid surface representing the air/Earth boundary. The sound speed and density profiles within the atmosphere are as shown in Figure 2, and the source is located at an altitude of 35 km. Acoustic field solutions are derived for source periods of 100 s, 200 s, and 300 s. There periods were chosen because, as indicated in Figure 3, the acoustic wavelength k_m , defined in Eq. (5), is approximately equal to $k = \omega/c$ at a period at 100s; at T = 200 s, k_m differs significantly from k, and at T = 300 s, the acoustic wavelength is predicted to have an imaginary value. The transmission loss solutions are shown in Figure 4. The results suggest that the low-frequency acoustic fields would undergo significant attenuation with increasing distance from the source. Note that these computations neglect the effects of gravity.

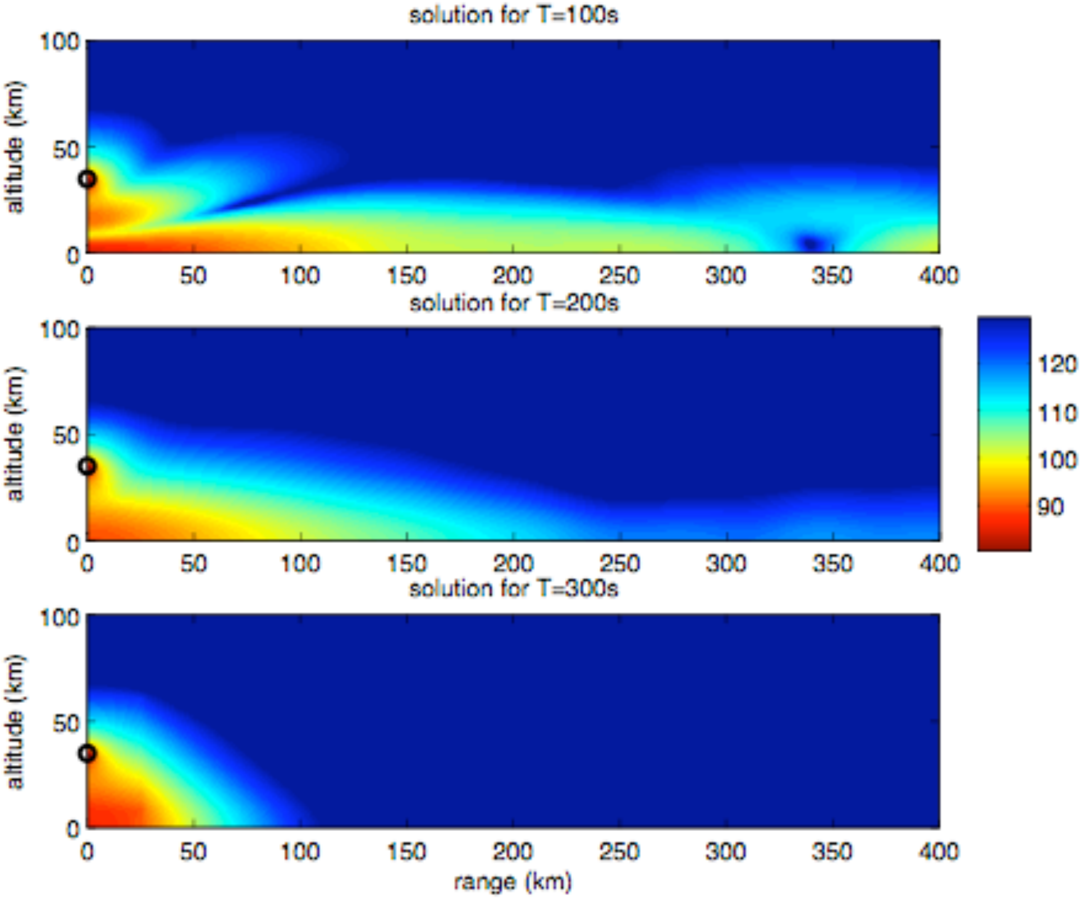

Figure 4. Finite difference transmission loss solutions for source periods of T = 100 s (top), T = 200 s (center) and T = 300 s (bottom). The model velocities and densities are as shown in Figure 2 (the density profile is given by the blue line in the center of Figure 2) and is terminated at the bottom by a rigid boundary. Absorbing boundary conditions are used to simulate a model with infinite extent with both increasing range and altitude. The color scale is identical for each plot.

FD Solution for Acousto-Gravity Propagation in the Atmosphere

An FD computation of acousto-gravity fields involves discretizing Eqs. (2). The density and pressure perturbations caused by the passage of a sound wave are computed simultaneously in this formulation so that the acousto-gravity formulation is more computationally intensive than is the acoustic-only solution. However, the model discretization in each coordinate direction scales with the acoustic wavelength—approximately 15 nodes per wavelength are required for an accurate solution. Acousto-gravity effects are significant only at very low frequencies—for wavelengths on the order of 100 km. Therefore, realistic atmospheric models for acousto-gravity effects have spatial scales on the order of several wavelengths in the z direction. Thus, the FD approach can handle acousto-gravity problems with reasonable computational efficiency.

CONCLUSIONS AND RECOMMENDATIONS

It was shown that the exponential decrease in atmospheric density plays an important role in acoustic propagation at very long periods. For periods of $T > 100$ s. the ratio of the acoustic wavelength to the period increases rapidly. At periods greater than $T = 300$ s, the acoustic wave equations predict that the wave number is imaginary; thus, acoustic propagation dissipates rapidly away from the source. However, these results ignore the effects of gravity. At very long periods, buoyancy effects become significant, as density perturbations caused by the passage of a sound wave play an increasingly important role. Including gravity in the FD equations reduces the stability of the algorithm; methods of increasing the stability are under investigation.

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