# 7: Deformation and Traction

The forces that act on the earth cause *deformation*, that is they cause the material in the earth to change shape. A force such as gravity can act directly upon a particle of material (a *body* force) or indirectly through interactions between neighboring particles. In the latter case the forces are transmitted between surfaces in contact, leading to the idea of surface *traction*, that is the force per unit area exerted on the surface of a particle.

As we will see below, two tensors are used to study traction and deformation, *stress* (which is related to traction) and *strain* (which is related to deformation). The words 'stress and strain' are commonly used almost synonymously with 'traction and deformation'. This usage not withstanding, it is important to remember that 'stress and strain' refer to convenient mathematical abstractions, 'traction and deformation' to physical processes.

## 7.1 TRACTION AND THE STRESS TENSOR

Traction is the force per unit area acting upon a surface. It is a vector field; its magnitude and direction can vary with position. Several surfaces of different orientations, each of which pass through the same point, need not have the same tractions. We therefore denote traction by $\mathbf{T}(\mathbf{x},\hat{n})$, where $\hat{n}$ is the surface normal (figure 7.1).

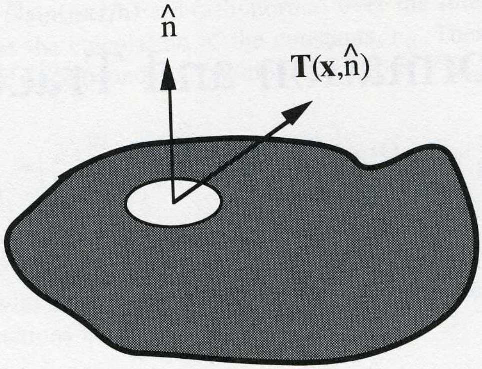


FIGURE 7.1 The traction, $\mathbf{T}(\mathbf{x},\hat{n})$, is the force per unit area acting on a small element of surface at position, x, and with surface normal, $\hat{n}$. The surface need not be on the exterior of the object. We can imagine subdividing the object into adjoining pieces, and considering the surface tractions that one piece exerts on the other.

The direction of the traction is typically oblique to the surface normal. The component parallel to the normal is called the *normal traction* and the component in the plane of the surface is called the *shear traction.*

While the tractions on several surfaces passing through the same point need not be identical, neither are they completely arbitrary. Consider, for instance, a pill box shaped volume (figure 7.2, left) where one flat face of the pill box has the normal $\hat{n}$, and the other the normal $-\hat{n}$. As the mass enclosed by the pill box is made indefinitely small by decreasing the distance between the two flat faces, the acceleration this mass experiences would increase indefinitely unless the tractions on the two opposing faces were equal and opposite. Thus $\mathbf{T}(\mathbf{x},\hat{n}) = -\mathbf{T}(\mathbf{x},-\hat{n})$. This argument can be generalized to a volume of arbitrary shape. The force acting on the surface of the volume, divided by the total surface area, must approach zero as the volume decreases to zero.

$$\lim_{V \to 0} \frac{\oiint_S \mathbf{T}(\mathbf{x},\hat{n})\, dS}{\oiint_S dS} = 0 \tag{7.1.1}$$

For simplicity, we consider a two-dimensional triangular volume (figure 7.2, right). Working in cartesian coordinates, we find that Equation 7.1.1 becomes:

$$\lim_{V \to 0} \frac{T_i(-\hat{z})\Delta x + T_i(-\hat{x})\Delta z + T_i(\hat{n})\,[(\Delta x)^2+(\Delta z)^2]^{1/2}}{\Delta x + \Delta z + [(\Delta x)^2+(\Delta z)^2]^{1/2}} = 0 \tag{7.1.2}$$

The components of $\hat{n}$ can be introduced into this expression by noting:

$$n_x = \frac{\Delta z}{[(\Delta x)^2+(\Delta z)^2]^{1/2}} \quad \text{and} \quad n_z = \frac{\Delta x}{[(\Delta x)^2+(\Delta z)^2]^{1/2}} \tag{7.1.3}$$

so that Equation 7.1.2 becomes:

$$\lim_{V \to 0} \frac{\left\{T_i(-\hat{z})n_z + T_i(-\hat{x})n_x + T_i(\hat{n})\right\}[(\Delta x)^2+(\Delta z)^2]^{1/2}}{\Delta x + \Delta z + [(\Delta x)^2+(\Delta z)^2]^{1/2}} = 0 \tag{7.1.4}$$

In order for the limit to be satisfied, the expression in the braces must be identically zero:

$$T_i(-\hat{z})n_z + T_i(-\hat{x})n_x + T_i(\hat{n}) = 0 \quad \text{or} \quad T_i(\hat{n}) = T_i(\hat{z})n_z + T_i(\hat{x})n_x \tag{7.1.5}$$

Here we have used the result, $\mathbf{T}(\mathbf{x},\hat{n}) = -\mathbf{T}(\mathbf{x},-\hat{n})$, between the first and second equations. The normal, $\hat{n}$, and traction, $\mathbf{T}$, are related by the tensor rule:

$$\begin{bmatrix} T_x(\hat{n}) \\ T_z(\hat{n}) \end{bmatrix} = \begin{bmatrix} T_x(\hat{x}) & T_x(\hat{z}) \\ T_z(\hat{x}) & T_z(\hat{z}) \end{bmatrix} \begin{bmatrix} n_x \\ n_z \end{bmatrix} \tag{7.1.6}$$

The traction on an arbitrarily oriented surface is a linear combination of the tractions on a set of coordinate surfaces. In three dimensions–a simple generalization of this two-dimensional example–the traction and surface normal each have three components and are related by:

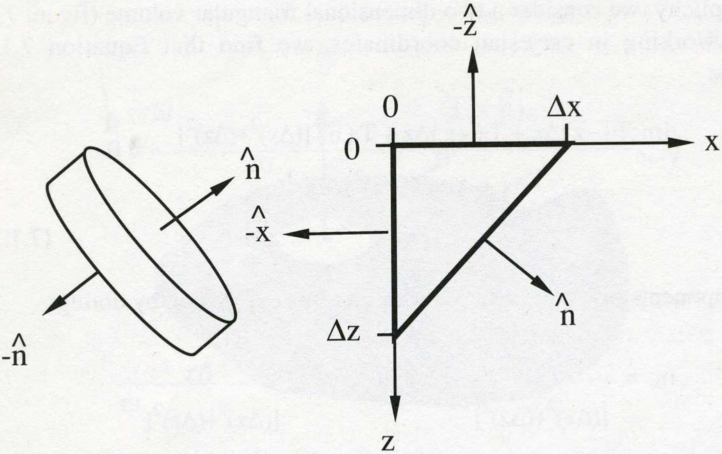


FIGURE 7.2 (left) A small, pill box shaped volume. Its two flat faces have the normals $\hat{n}$ and $-\hat{n}$. (right) A two-dimensional, triangular volume. These volumes are used to derive the stress tensor.

$$\begin{bmatrix} T_x(\hat{n}) \\ T_y(\hat{n}) \\ T_z(\hat{n}) \end{bmatrix} = \begin{bmatrix} T_x(\hat{x}) & T_x(\hat{y}) & T_x(\hat{z}) \\ T_y(\hat{x}) & T_y(\hat{y}) & T_y(\hat{z}) \\ T_z(\hat{x}) & T_z(\hat{y}) & T_z(\hat{z}) \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \quad \text{or} \quad T_i(\hat{n}) = T_i(\hat{x}_j)\, n_j \tag{7.1.7}$$

where we are again limiting our discussion to cartesian coordinates. The quantity $\tau_{ij} = T_i(\hat{x}_j)$ is called the *stress tensor*. Its elements are just the components of the tractions on a set of coordinate surfaces. Given the stress tensor, we can easily compute the traction on a surface of arbitrary orientation. The stress tensor can vary with position and time; we will often write it as $\tau_{ij}(\mathbf{x},t)$.

We shall also assume that the torque, **t**, due to tractions acting upon the surface of a volume also vanishes if the volume is made very small. We find that the cartesian coordinates of the torque are (see Equation 4.1.8):

$$t_i = \oiint_S \varepsilon_{ijk} x_j T_k(\mathbf{x},\hat{n})dS = \oiint_S \varepsilon_{ijk} x_j \tau_{km} n_m dS = \iiint_V \varepsilon_{ijk} x_{j,m} \tau_{km}\, dV \rightarrow 0 \tag{7.1.8}$$

Here we have applied Gauss' theorem to the second integral and have assumed that the traction varies slowly with position so that we can ignore its spatial derivatives. Noting that $x_{j,m} = \delta_{jm}$, we have:

$$t_i = \iiint_V \varepsilon_{ijk}\tau_{kj}\, dV \rightarrow 0 \tag{7.1.9}$$

This equation can be satisfied for arbitrary V only when $\varepsilon_{ijk}\tau_{kj} = 0$, that is the stress tensor is symmetric. Finally we note that if a tensor is symmetric in one coordinate system, then it is symmetric in all coordinate systems, so that the symmetry is not limited to cartesian coordinates.

One important property of the traction, and hence of corresponding components of the stress tensor, is that they are continuous functions of position (except at points where singular forces, such as delta functions, are also acting). This behavior can be understood by referring to the pill box of figure 7.2. If the tractions on the two flat faces of the pill box were different in the limit of the distances between the faces becoming very small, then the mass within the box would experience infinite acceleration.

The stress tensor can be conveniently divided into two parts, an isotropic part proportional to the identity matrix and the remaining, *deviatoric*, part:

$$\tau_{ij} = -p\,\delta_{ij} + d_{ij} \quad \text{where } p = -\frac{1}{3}\tau_{kk} \tag{7.1.10}$$

The proportionality constant, p, is called the *pressure*. It represents the mean, inward directed force that acts to compress a small particle. In contrast, the deviatoric part, $d_{ij}$, acts to shear a small volume. Pressure is of particular relevance to the study of perfectly elastic fluids, since they cannot support deviatoric stresses.

## 7.2 DEFORMATION AND STRAIN

When a body is deformed, the particles within it move with respect to their initial, reference, positions. This change in position is called *displacement*, and is quantified by the vector $\mathbf{u}(\mathbf{x},t)$. Since the body is deforming, we need to be very careful in specifying exactly which particles we are referring to. Should $\mathbf{u}(\mathbf{x},t)$ be taken to mean the displacement at time, t, of the particle that was at position $\mathbf{x}$ at the reference time? Or should it be taken to mean the displacement at time t of the particle at position $\mathbf{x}$ at time t? These may be two completely different particles. In fact, both interpretations are perfectly sensible and have their application.

The first, or *Lagrangian*, interpretation, tracks individual particles. The relationship between particle displacement, $\mathbf{u}$, and particle velocity, $\mathbf{v}$, is very simple (at least in cartesian coordinates): $\mathbf{v} = \partial\mathbf{u}/\partial t$. On the other hand, spatial derivatives are complicated, since ordinarily it is the derivative with respect to the current position (and not the reference position) that is of interest.

The second, or, *Eulerian*, interpretation, considers whatever particle happens to be at position $\mathbf{x}$ at a given time. Spatial derivatives are thus very easily interpreted. On the other hand, time derivatives are more complicated. For instance, if we define the Eulerian velocity, $\mathbf{v}(\mathbf{x},t)$, to mean the velocity of whatever particle is at $\mathbf{x}$ at time t, the acceleration of this particle is not $\partial\mathbf{v}/\partial t$, since the particle is only at that position instantaneously. Instead, the acceleration, $\mathbf{a}(\mathbf{x},t)$, is computed by differencing the current velocity of the particle with the velocity at the position and time where it will be an instant later:

$$\mathbf{a}(\mathbf{x},t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(\mathbf{x}+\mathbf{v}\Delta t, t+\Delta t) - \mathbf{v}(\mathbf{x},t)}{\Delta t} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla\mathbf{v})\bullet\mathbf{v} \tag{7.2.1}$$

This combination of derivatives is often called the *material derivative*.

The deformation problems that arise in geophysics can be divided into two general categories: Flow problems, such as convection in the mantle, where the displacements of particles are large and tend to increase with time; and vibration problems, such as seismic wave propagation, where the displacements are small and tend to oscillate about zero. The Eulerian viewpoint is the most useful for flow problems, since the reference position of the particles tends not to have any particular significance (what should be used as the reference position of a particle of water in the Hackensack River?). In fact, particle velocity is usually used as the primary dependent variable instead of displacement. On the other hand, vibration problems generally involve such small displacements (compared to the wavelength of the vibrations) that the two viewpoints are essentially identical.

Regardless of whether we are using an Eulerian or Lagrangian viewpoint, displacement does not necessarily imply deformation. For instance, if all the particles in an object are uniformly displaced, then the object has merely been moved, or *translated*, from its reference position. Similarly, rotating an object leads to displacement but not deformation. In order to quantify the deformation associated with a particular displacement, we must consider the differential displacement between two neighboring points within the object. For the sake of clarity, we adopt the Lagrangian viewpoint and consider two particles initially at reference positions at $\mathbf{x}$ and $\mathbf{x} + \Delta\mathbf{x}$, respectively (figure 7.3). After the displacement occurs, the two particles are separated by $\Delta\mathbf{x} + \Delta\mathbf{u}$, where $\Delta\mathbf{u} = \mathbf{u}(\mathbf{x} + \Delta\mathbf{x}) - \mathbf{u}(\mathbf{x})$. Since the separation distance is presumed small, we can apply Taylor's theorem:

$$\Delta u_i = u_i(\mathbf{x}+\Delta\mathbf{x}) - u_i(\mathbf{x}) = \frac{\partial u_i}{\partial x_j}\Delta x_j \tag{7.2.2}$$

The question of whether deformation has occurred is equivalent to asking whether the separation between the particles has changed, that is whether the $|\Delta\mathbf{x}+\Delta\mathbf{u}|$ differs from $|\Delta\mathbf{x}|$. Evidently, the change in length is related to the properties of the derivatives $\partial u_i/\partial x_j$. As we will see, it is useful to break this derivative up into its symmetric and antisymmetric parts:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] + \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right] = \varepsilon_{ij} + \Omega_{ij} \tag{7.2.3}$$

The new separation between the particles is then:

$$|\Delta\mathbf{x}+\Delta\mathbf{u}| = \left[\ (\Delta x_i+\varepsilon_{ij}\Delta x_j+\Omega_{ij}\Delta x_j)\ (\Delta x_i+\varepsilon_{ik}\Delta x_k+\Omega_{ik}\Delta x_k)\ \right]^{1/2} \tag{7.2.4}$$

We now multiply the two quantities in parenthesis, discarding terms involving products of derivatives (that is products of $\varepsilon_{ij}$ and $\Omega_{ij}$), which are by assumption very small. The quantity $2\Omega_{ij}\Delta x_i\Delta x_j$ is also zero, since it involves the contraction of an antisymmetric quantity with a symmetric quantity. Thus:

$$|\Delta\mathbf{x}+\Delta\mathbf{u}| = \left[\Delta x_i\Delta x_i + 2\varepsilon_{ij}\Delta x_j\Delta x_k\right]^{1/2} \approx |\Delta\mathbf{x}|\left(1 + \frac{\varepsilon_{ij}\Delta x_j\Delta x_k}{|\Delta\mathbf{x}|^2}\right) \tag{7.2.5}$$

Here we have used Taylor's theorem to approximate the square root. The quantity $\nu_i = \Delta x_i/|\Delta\mathbf{x}|$ is just a unit vector connecting the reference positions of the two particles. Thus we can write:

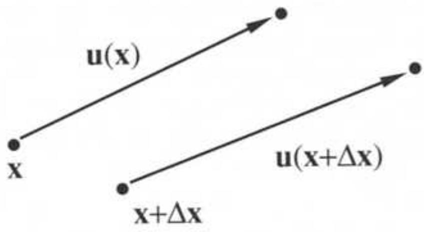


FIGURE 7.3 At the reference time, two particles are located at positions $\mathbf{x}$ and $\mathbf{x} + \Delta\mathbf{x}$, respectively. At a later time they are displaced to positions $\mathbf{x}+ \mathbf{u}(\mathbf{x})$ and $\mathbf{x}+\Delta\mathbf{x}+\mathbf{u}(\mathbf{x}+\Delta\mathbf{x})$, respectively. If the displacement involves deformation, the distance separating the two particles will have changed.

$$\frac{\text{new separation - old separation}}{\text{old separation}} = \varepsilon_{ij}\nu_i\nu_j \tag{7.2.6}$$

The deformation of the body is related to the symmetric part of the derivative, which is called the *strain tensor*, $\varepsilon_{ij}$. A similar calculation demonstrates that the rotation of the body is related only to $\Omega_{ij}$, the *rotation tensor*. The volume change related to the strain can be computed by considering a small cube with sides of length, $\Delta x$, aligned along the coordinate directions. After being strained, the volume changes from $(\Delta x)^3$ to:

$$(\Delta x)^3(1+\varepsilon_{11})(1+\varepsilon_{22})(1+\varepsilon_{33}) \approx (\Delta x)^3(1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

The quantity $\varepsilon_{ii}$ is often called the *volumetric strain*. Similarly, the part of the strain not associated with a volume change, $\varepsilon_{ij}-\varepsilon_{kk}\delta_{ij}/3$, is often called the *shear strain*.

Unless an object cracks or faults during the course of deformation, the displacement field is everywhere continuous. The strain, on the other hand, is a derivative of displacement, and may be discontinuous in regions where there are sharp changes in the material properties (for example, at layer boundaries). As we will see in subsequent chapters, the continuity of displacement and stress make them the logical choice as the primary variables for studying deformation.

Since we have performed this derivation using Lagrangian displacements, a non-zero strain tensor at $\mathbf{x}$ means that the material that was originally at $\mathbf{x}$ has deformed. Had the derivation been performed using Eulerian displacements, the formula for strain would be the same, but the

$$\frac{\partial}{\partial t}(\rho v_i) = \rho \frac{\partial}{\partial t} v_i + v_i \frac{\partial \rho}{\partial t} = -v_i(\rho v_j)_{,j} - \rho v_j v_{i,j} + f_i + \tau_{ij,j} \tag{7.3.8}$$

The law of conservation of mass (Equation 7.3.3) can be used to eliminate one term from each side of the equation, leaving:

$$\rho \frac{\partial}{\partial t} v_i + \rho v_j v_{i,j} = \tau_{ij,j} + f_i \quad \text{or} \quad \rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v})\bullet\mathbf{v}\right) = \nabla\bullet\tau + \mathbf{f} \tag{7.3.9}$$

The momentum equation is very important to the study of deformation, since it constitutes one of the two parts of *an equation of motion*. The other part, to be discussed in the next section, is a *constitutive law*, that is, a relationship between the stress tensor and the velocity field. Together with such a law, Equation 7.3.9 is a partial differential equation in a single unknown, the velocity, **v**.

Equation 7.3.9 can be simplified somewhat in vibration problems, where we assume that the strains are small and that the wavelength of the vibrations are very long compared to the total displacements. Then $v_i \approx \partial u_i/\partial t$, and $v_i v_{i,j} \approx 0$, so that:

$$\rho \frac{\partial^2}{\partial t^2} u_i = \tau_{ij,j} + f_i \tag{7.3.10}$$

## 7.4 CONSTITUTIVE LAWS

### 7.4.1 Linear Elasticity

An elastic material is one in which the deformation can be undone by removing the force. The linear spring is a well known idealization of an elastic substance. It elongates linearly with the applied force, and snaps back to its original length when the force is released. This behavior is characterized by Hooke's law: $f = k\Delta x$, where k is the spring constant and $\Delta x$ is the elongation. No energy is expended by stretching and compressing a spring; the energy stored in the spring as it is stretched is exactly returned when it springs back.

Solid materials can be idealized as being linear and elastic, as long as the amount of deformation is small. The stress is then proportional to the strain. Since stress and strain are second order tensors, the proportionality constant is a fourth order tensor, $c_{ijpq}$:

$$\tau_{ij} = c_{ijpq}\varepsilon_{pq} \tag{7.4.1}$$

While the *Hooke's law tensor*, $c_{ijpq}$, has 81 components, various symmetry requirements (such as $\varepsilon_{ij} = \varepsilon_{ji}$, $\tau_{ij} = \tau_{ji}$, and others we will not discuss here) reduce the number of independent elements to 21 for a general anisotropic material. If we wanted to study the deformation of a single anisotropic crystal (say turquoise, which is triclinic), we would need an equation of motion that involves 21 empirically measured parameters. Fortunately, the earth is nearly (but not exactly) isotropic, because the many crystals that compose it, while themselves anisotropic, tend to be randomly oriented. The isotropic Hooke's law tensor turns out to have only two independent *Lamé* parameters, $\lambda$, and the *rigidity*, $\mu$:

$$c_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu\left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}\right) \tag{7.4.2}$$

Stress is related to strain by:

$$\tau_{ij} = c_{ijpq}\,\varepsilon_{pq} = \lambda\delta_{ij}\varepsilon_{pp} + 2\mu\varepsilon_{ij} \tag{7.4.3}$$

or

$$\begin{aligned}
\tau_{xx} &= (\lambda+2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} + \lambda\varepsilon_{zz}\\
\tau_{yy} &= \lambda\varepsilon_{xx} + (\lambda+2\mu)\varepsilon_{yy} + \lambda\varepsilon_{zz}\\
\tau_{zz} &= \lambda\varepsilon_{xx} + \lambda\varepsilon_{yy} + (\lambda+2\mu)\varepsilon_{zz}\\
\tau_{xy} &= \tau_{yx} = 2\mu\varepsilon_{xy}\\
\tau_{xz} &= \tau_{zx} = 2\mu\varepsilon_{xz}\\
\tau_{yz} &= \tau_{zy} = 2\mu\varepsilon_{yz}
\end{aligned} \tag{7.4.4}$$

The stress-strain relationship is even simpler in a perfectly elastic fluid, which cannot support a shear stress. The pressure completely describes the state of stress and is proportional to the volumetric strain:

$$p = -\lambda\, \varepsilon_{kk} \tag{7.4.5}$$

### 7.4.2 The Incompressible, Viscous Fluid

At the other extreme from an elastic material is a viscous material, which permanently deforms in response to an applied stress. The well-known dash pot of elementary physics textbooks typifies such a situation. A force causes a paddle immersed in a fluid-filled pot to move at a constant rate. This idea can be extended to idealized materials within the earth by assuming that they are incompressible (that is, their volumetric strain is always zero) and that their deviatoric stress is proportional to the shear strain rate:

$$\tau_{ij} + p\delta_{ij} = 2\nu'\dot{\varepsilon}_{ij} \tag{7.4.6}$$

The proportionality factor, $\nu'$, is called the viscosity. Fluids that obey this equation are said to be incompressible, *Newtonian* fluids. Studies of convection within the earth often assume viscous flow occurs within the mantle. Laboratory experiments, however, indicate that the earth's mantle is not Newtonian. Stress is related to strain rate by a power law. Nevertheless, Newtonian fluids provide a useful idealization in certain cases.

### 7.4.3 The Maxwell Solid

The ideas underlying elasticity and viscosity can, of course, be combined. Most real materials behave somewhere between the two extremes: over short time scales they respond elasticly, though with some slight permanent deformation, and over long time scales they flow viscously, but with some slight compression. The simplest relationship combining these ideas is the *Maxwell Solid* constitutive law:

$$\dot{\tau}_{ij} + \frac{\mu}{\nu'}\left(\tau_{ij}+p\delta_{ij}\right) = \lambda\dot{\varepsilon}_{kk}\delta_{ij} + 2\mu\dot{\varepsilon}_{ij} \tag{7.4.7}$$

# 9: Seismology and Elastic Wave Propagation

In this chapter we will discuss the propagation of vibrations, or *elastic waves*, through solids. *Seismology* is the application of this material to the study of the structure of the earth, and to earthquakes, which are a major source of elastic waves.

The material in this chapter relies heavily on the development of acoustic wave propagation in chapter 8. Its main purpose is to bring out the similarities and differences between the acoustic and elastic cases. For this reason, and because of the inherent complexity of elastic waves, more of the derivations are omitted than in other chapters of this book.

## 9.1 ELASTIC WAVES IN HOMOGENEOUS, ISOTROPIC MEDIA

### 9.1.1 The Elastic Wave Equation

The elastic wave equation is formed by combining the momentum equation, which relates time derivatives of displacement to spatial deriviatives of the stress tensor (equation 7.3.10), with the elastic constitutive law (which related the stress tensor to spatial derivatives of displacement. The result is a single equation in displacement. We perform the derivation in Cartesian coordinates, where the momentum equation is $\rho \ddot{u}_i = \tau_{ij,j} + f_i$, for a homogeneous, isotropic material, in which stress and strain are related by $\tau_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$ (equation 7.4.3):

$$\rho\ddot{u}_i = (\lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij})_{,j} + f_i = (\lambda\delta_{ij}u_{k,k} + \mu u_{i,j} + \mu u_{j,i})_{,j} + f_i$$

$$= \lambda u_{k,ki} + \mu u_{i,jj} + \mu u_{j,ji} + f_i = (\lambda + \mu)u_{k,ki} + \mu\, u_{i,jj} + f_i \tag{9.1.1}$$

In vector notation, the elastic wave equation is:

$$\rho\ddot{\mathbf{u}} = (\lambda + \mu)\, \nabla(\nabla\bullet\mathbf{u}) + \mu\, \nabla^2\mathbf{u} + \mathbf{f} = (\lambda + 2\mu)\, \nabla(\nabla\bullet\mathbf{u}) - \mu\, \nabla \times \nabla \times \mathbf{u} + \mathbf{f} \tag{9.1.2}$$

Note that we have used the identity, $\nabla^2\mathbf{u} = \nabla(\nabla\bullet\mathbf{u}) - \nabla \times \nabla \times \mathbf{u}$ in the second equality of equation 9.1.12.

### 9.1.2 Plane Waves

We first examine whether the elastic wave equation possesses, in the absence of forces, plane wave solutions similar to those of the acoustic wave equation (see section 8.2). The major difference between the two equations is that the acoustic equation is a scalar wave equation in pressure, while the elastic equation, as it is written in equation 9.1.2, is a vector wave equation in displacement. While we could recast equation 9.1.2 as an equation in stress, we would not be able to manipulate it into a scalar equation, since, unlike a fluid, a solid can support both shear and normal stresses. We must therefore generalize the notion of a harmonic plane wave to a *vector plane wave* of frequency, $\omega$, which has the form:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{A}\exp\{i\mathbf{k}\bullet\mathbf{x} - i\omega t\} \tag{9.1.3}$$

Here the vector, $\mathbf{A}$, called the *polarization vector*, specifies not only the overall amplitude of the displacement, but also the direction along which it occurs. The vector, $\mathbf{k}$, is the usual wavenumber vector that is parallel to the direction of propagation of the plane wave. Inserting equation 9.1.3 into the force-free elastic wave equation (equation 9.1.1 with $f_i = 0$) yields:

$$-\rho\omega^2 A_i \exp\{i\mathbf{k}\bullet\mathbf{x} - i\omega t\} =$$

$$-(\lambda + \mu)\, A_k k_k k_i \exp\{i\mathbf{k}\bullet\mathbf{x} - i\omega t\} - \mu\, A_i k_j k_j \exp\{i\mathbf{k}\bullet\mathbf{x} - i\omega t\} \tag{9.1.4}$$

After canceling a factor of $-\exp\{i\mathbf{k}\cdot\mathbf{x} - i\omega t\}$ from this equation, we are left with a relationship between the polarization and wavenumber vectors:

$$\rho\omega^2 A_i = (\lambda + \mu)\, A_k k_k k_i + \mu\, A_i k_j k_j \tag{9.1.5}$$

This equation has two distinct classes of solutions.

The first class of solutions occurs when the polarization vector, **A**, is parallel to the wavenumber vector, **k**, say **A** = C**k** where C is a constant. Then equation 9.1.5 becomes:

$$\rho\, \omega^2\, k_i = (\lambda+\mu)\, |k|\, |k|\, k_i + \mu\, |k|\, |k|\, k_i \quad \text{or} \quad \omega^2 / |k|^2 = (\lambda + 2\mu) / \rho \tag{9.1.6}$$

Notice that the factor of C cancels from the equation, indicating that the overall amplitude of the plane wave is arbitrary. This plane wave propagates at a phase velocity of $\omega/|\mathbf{k}| = [(\lambda + 2\mu)/\rho]^{1/2}$. It has a *longitudinal* particle motion, meaning that its particle motion is always parallel to its propagation direction. The divergence of the displacement, $u_{i,i} = iCk_ik_i\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega t)$, is non-zero and the curl, $\varepsilon_{ijk}u_{k,j} = iC\varepsilon_{ijk}k_jk_k\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega t)$, is zero (since the cross product of parallel vectors is zero), indicating that these waves consist of compressive motions that involve no twisting. They are called *compressional* waves (figure 9.1, top) and their propagation velocity is given the symbol, $\alpha = [(\lambda + 2\mu)/\rho]^{1/2}$.

The second class of solutions occurs when the polarization vector, **A**, is perpendicular to the wavenumber vector, **k**. Then equation 9.1.5 becomes:

$$\rho\, \omega^2\, A_i \;=\; \mu\;\; A_i\, k_j\, k_j \quad \text{or} \quad \omega^2 / |k|^2 = \mu / \rho \tag{9.1.7}$$

This plane wave propagates at a phase velocity of $\omega/|\mathbf{k}| = [\mu/\rho]^{1/2}$. It has *transverse* particle motion, meaning that its particle motion is always perpendicular to its propagation direction. The divergence of the displacement, $u_{i,i}=iA_ik_i\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega t)$, is zero (since the dot product of perpendicular vectors is zero) and the curl, $\varepsilon_{ijk}u_{k,j} = i\varepsilon_{ijk}A_kk_j\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega t)$ is non-zero, indicating that these waves consists of twisting motions that involve no compression. They are called *shear* waves (figure 9.1, bottom), and their propagation velocity is given the symbol, $\beta = [\mu/\rho]^{1/2}$.

The vertical wavenumber, $k_z$, of a compressional and a shear wave differ, reflecting the different propagation velocities of the two waves:

$$k_{z\alpha} = i\gamma_\alpha = (\omega^2/\alpha^2 - k_x{}^2)^{1/2} \quad \text{and} \quad k_{z\beta} = i\gamma_\beta = (\omega^2/\beta^2 - k_x{}^2)^{1/2} \tag{9.1.8}$$

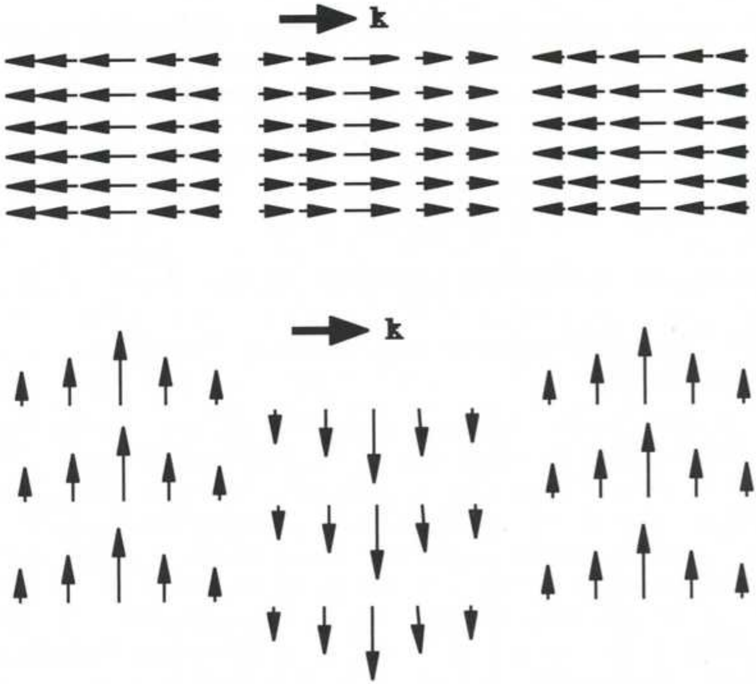


FIGURE 9.1 (Top) The displacement of a plane compressional wave is longitudinal, that is, parallel to the direction of propagation of the wave. (Bottom) The displacement of a plane shear wave is transverse, that is, perpendicular to the direction of propagation.

Here we have labeled the vertical wavenumbers of the compressional and shear waves with their velocities, $\alpha$ and $\beta$, respectively. As in the acoustic case, the waves can be either propagating or exponentially decaying (evanescent) in the z direction. When they are evanescent, the vertical wavenumbers can conveniently be written in terms of the real quantities, $\gamma_\alpha$ and $\gamma_\beta$. Note that at any given frequency, $\omega$, the compressional wave becomes evanescent at a smaller wavenumber than the shear wave, since $\alpha > \beta$ (in many rocks, $\lambda \approx \mu$, so that $\alpha/\beta = \sqrt{3}$). In the propagating case, the angle that the propagation direction makes with the vertical also differs between compressional and shear waves:

$$k_x = \frac{\omega \sin(i)}{\alpha} = \frac{\omega \sin(j)}{\beta} \tag{9.1.9}$$