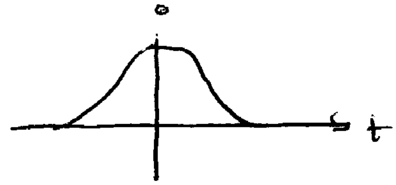
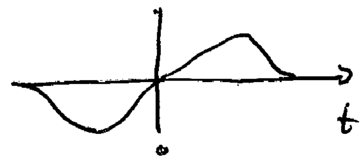
Fourier Representations

symmetric function $\overset{S}{f}(t)$



$\overset{S}{f}(t) = \overset{S}{f}(-t)$

antisymmetric fcn $\overset{A}{f}(t)$



$\overset{A}{f}(t) = -\overset{A}{f}(-t)$

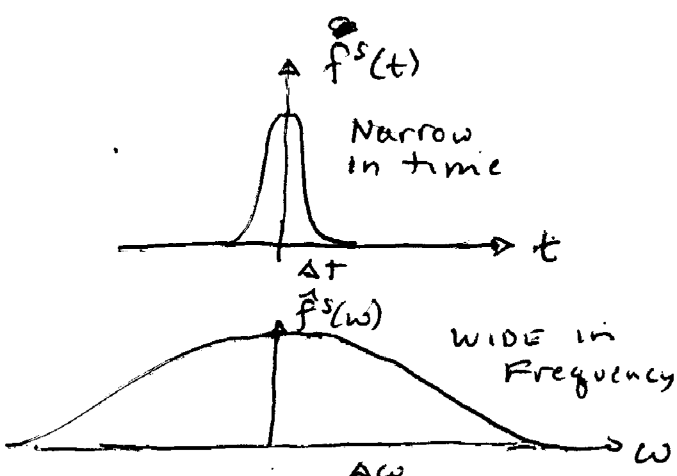
Idea construct function from sum of cosines (symmetric case) or sines (antisymmetric case) of all possible frequencies

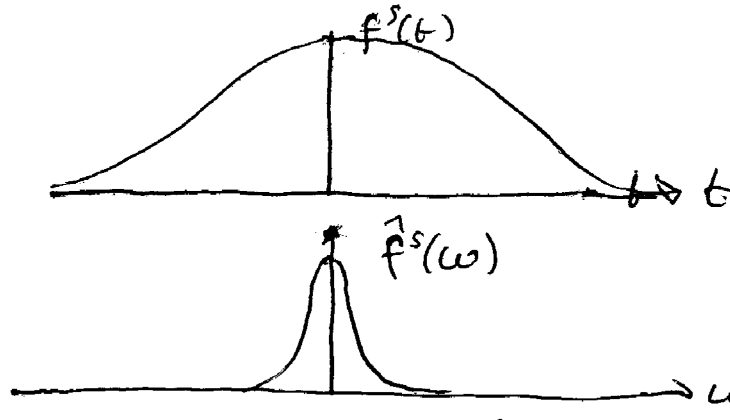
$$\overset{S}{f}(t) = \int_{-\infty}^{\infty} \hat{f}^{S}(\omega) \cos(-\omega t)\, d\omega$$

$$f^{A}(t) = \int_{\infty}^{\infty} \hat{f}^{A}(\omega) \sin(-\omega t)\, d\omega$$

$\hat{f}(\omega)$: amplitude of component sinusoid
"fourier components"

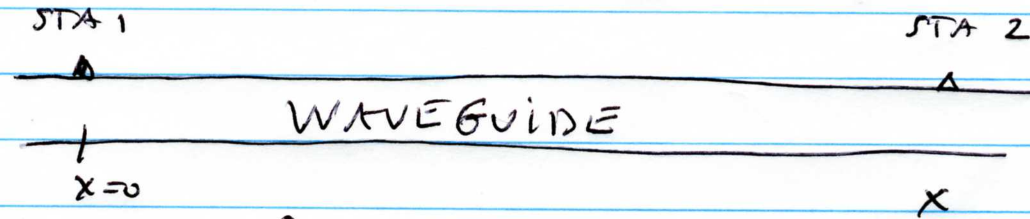
$\hat{f}^{2}(\omega)$: "Power" of component sinusoid

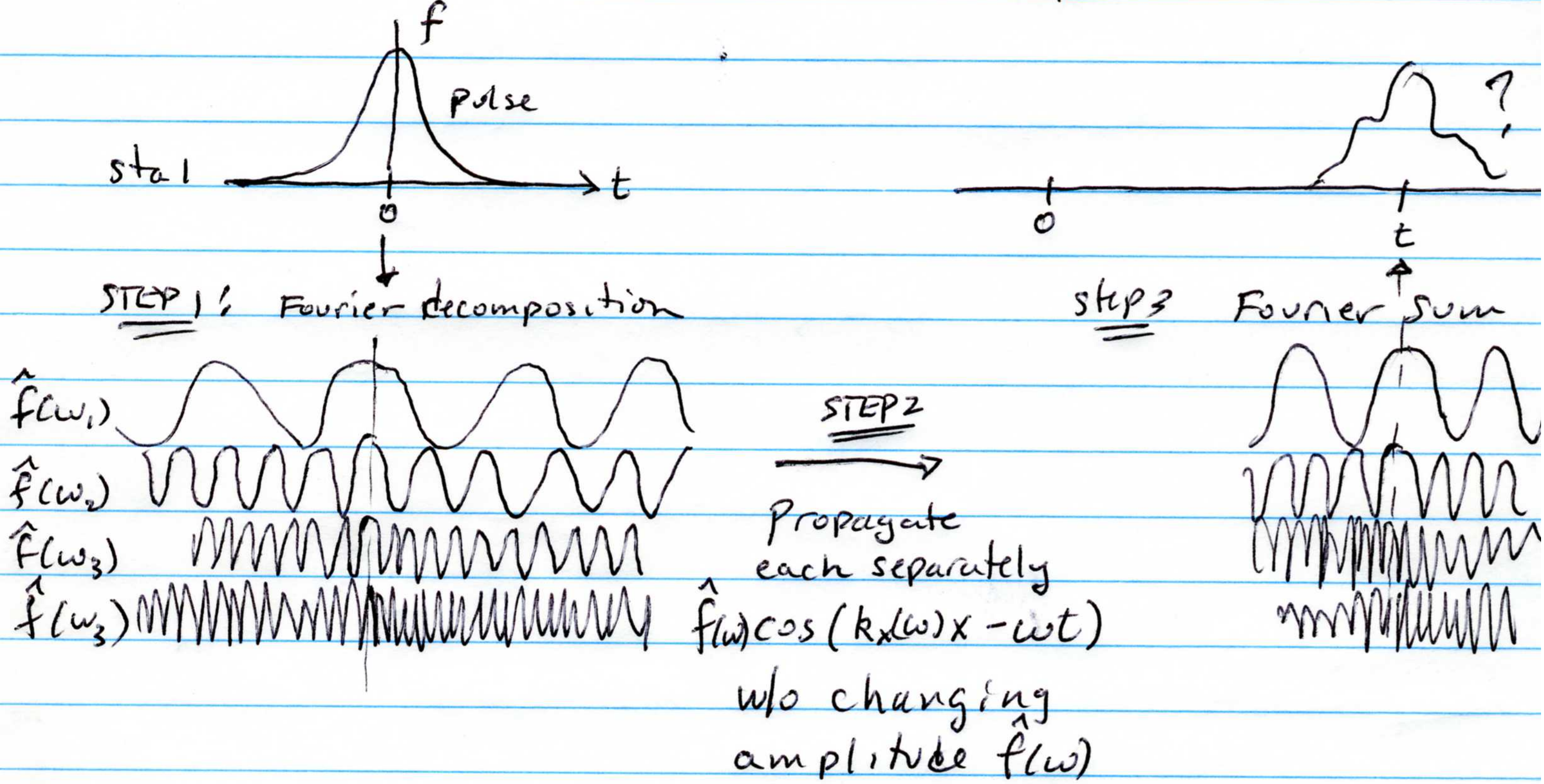




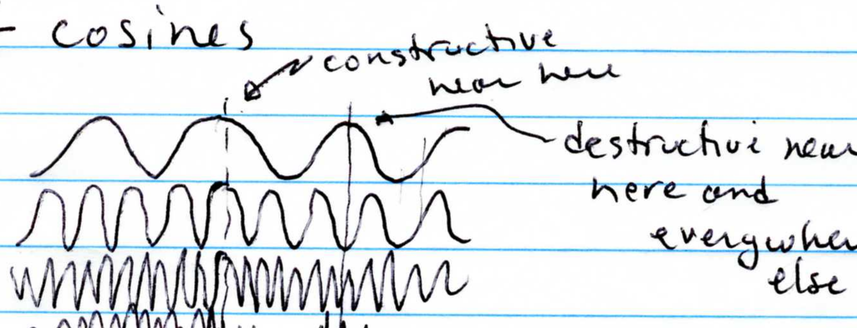
Heisenberg Uncertainty $\Delta T \Delta \omega = 2\pi$

evolution of surface waves





only get something when There is constructive interference of cosines



Condition for constructive interference:
phase of cosine <u>not</u> rapidly changing with frequency; peaks of different frequency waves <u>line up</u>. Phase is stationary for some bro

$$\cos(\underbrace{k_x(\omega)x - \omega t})$$

phase must not vary with frequency, at least for some finite range of frequencies. Phase stationary.

$$\frac{d}{d\omega}\left[k_x(\omega)x - \omega t\right] = 0$$

usual formula for a stationary point

For what frequency, ω_s, does $\frac{d}{d\omega}\left[k_x(\omega)x - \omega t\right]$ equal zero.

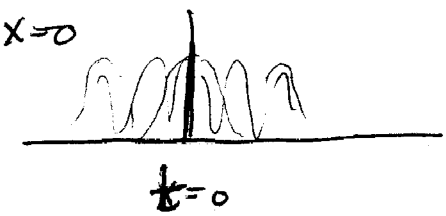
$$\left.\frac{dk_x(\omega)}{d\omega}\right|_{\omega=\omega_s} x = t \quad \text{or} \quad \frac{d\omega}{dk} = \frac{x}{t}$$

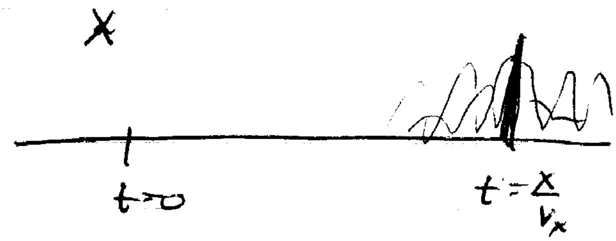
implicit equation for ω_s.

Non-Dispersive Case $k_x = \frac{\omega}{V_x}$ $V_x = \text{const}$

$\frac{d}{d\omega}\left(\frac{\omega x}{V_x} - \omega t\right) = 0$ implies $\frac{x}{V_x} - t = 0$

or stationary for entire range of frequencies ω_s = anything, but only at the time $t = \frac{x}{V}$.





whatever cosines lined up at $x=0, t=0$

also line up at x, $t=\frac{x}{V}$.

So pulse is <u>exactly transported</u> from $(x=0, t=0)$ to $(x, t=\frac{x}{V_x})$

pulse moves exactly at phase velocity, V_x.

But a dispersive material is more complicated ...

Dispersive case $\frac{d}{d\omega}\left(k_x(\omega)x - \omega t\right) = 0$

implies $\frac{dk_x(\omega)}{d\omega} = \frac{t}{x}$ and defines an

ω_s that solves this equation for a particular x (seismogram location) and time t (time in seismogram).

Now approximate the phase of the cosine by a Taylor series (keeping only leading terms

$$\cancel{k_x(\omega)}\quad k_x(\omega)x - \omega t \approx k_x(\omega_s)x - \omega_s t + 0 + \left.\frac{d^2k_x}{d\omega^2}\right|_{\omega=\omega_s}(\omega-\omega_s$$

and substitute into fourier sum

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega)\cos\left(k_x(\omega)x - \omega t\right)d\omega$$

$$\approx \int_{-\infty}^{+\infty} \hat{f}(\omega_s)\cos\Big(\underbrace{k_x(\omega_s)x - \omega_s t}_{a} + \underbrace{\frac{1}{2}\left.\frac{d^2k_x}{d\omega^2}\right|_{\omega=\omega_s}(\omega-\omega_s)^2 x}_{b}\Big)d\omega$$

now use $\cos(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$

$$\approx \hat{f}(\omega_s)\cos\left(k_x(\omega_s)x - \omega_s t\right)\int^{\infty}\sin\left(\left.\frac{d^2k_x}{d\omega^2}\right|(\omega-\omega_s)^2 x\right)d\omega$$

let's examine $\cos(k_x(\omega_s)x - \omega_s t)$ term.

at seismometer location x and time in seismogram t, There is a frequency ω_s that satisfies $d\omega/dk = \frac{x}{t}$.

near that (x,t) The seismogram is oscillatory with frequency ω_s.

all x's and t's with the same ratio, $\frac{x}{t} = \frac{d\omega}{dk}$ will have this frequency pulse shape.

So the osscilation of frequency ω_s appears to move at speed $d\omega/dk$ = group velocity $= u$

phase velocity $= V(\omega) = \omega/k(\omega)$

group velocity $= U(\omega) = \frac{d\omega}{dk}$

Note $u = v\left[1 - \frac{dv}{d\omega}\frac{\omega}{v}\right]^{-1}$ since

$u = \frac{d\omega}{dk} = \frac{d}{dk}(vk) = \frac{dc}{dk}k + v = \frac{dv}{d\omega}\frac{d\omega}{dk}\frac{\omega}{v} + v = \frac{dv}{d\omega}u\frac{\omega}{v} + v$

$u\left[1 - \frac{dv}{d\omega}\right.$