Solid Earth Dynamics

Bill Menke, Instructor

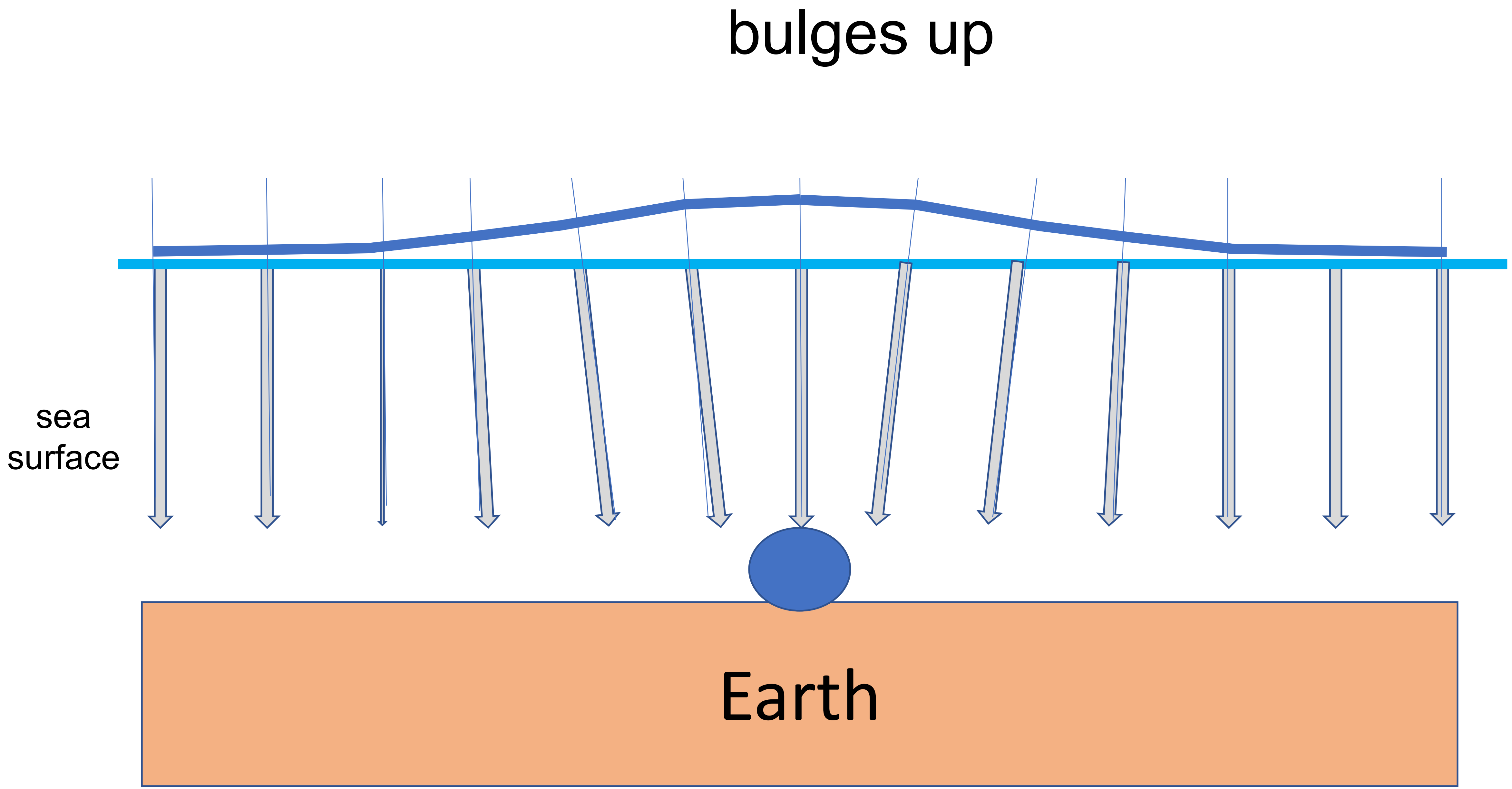
Lecture 8

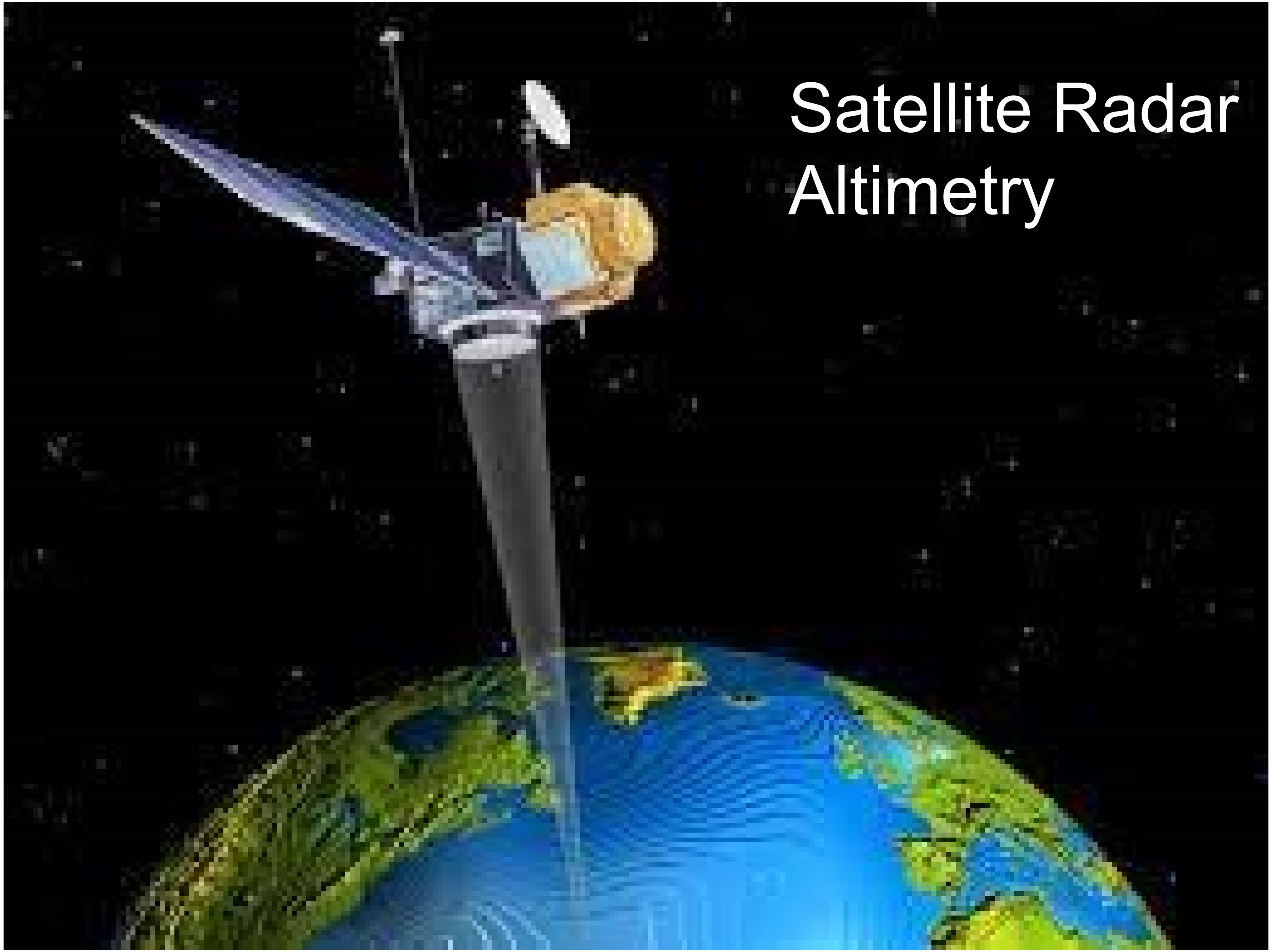
Today:

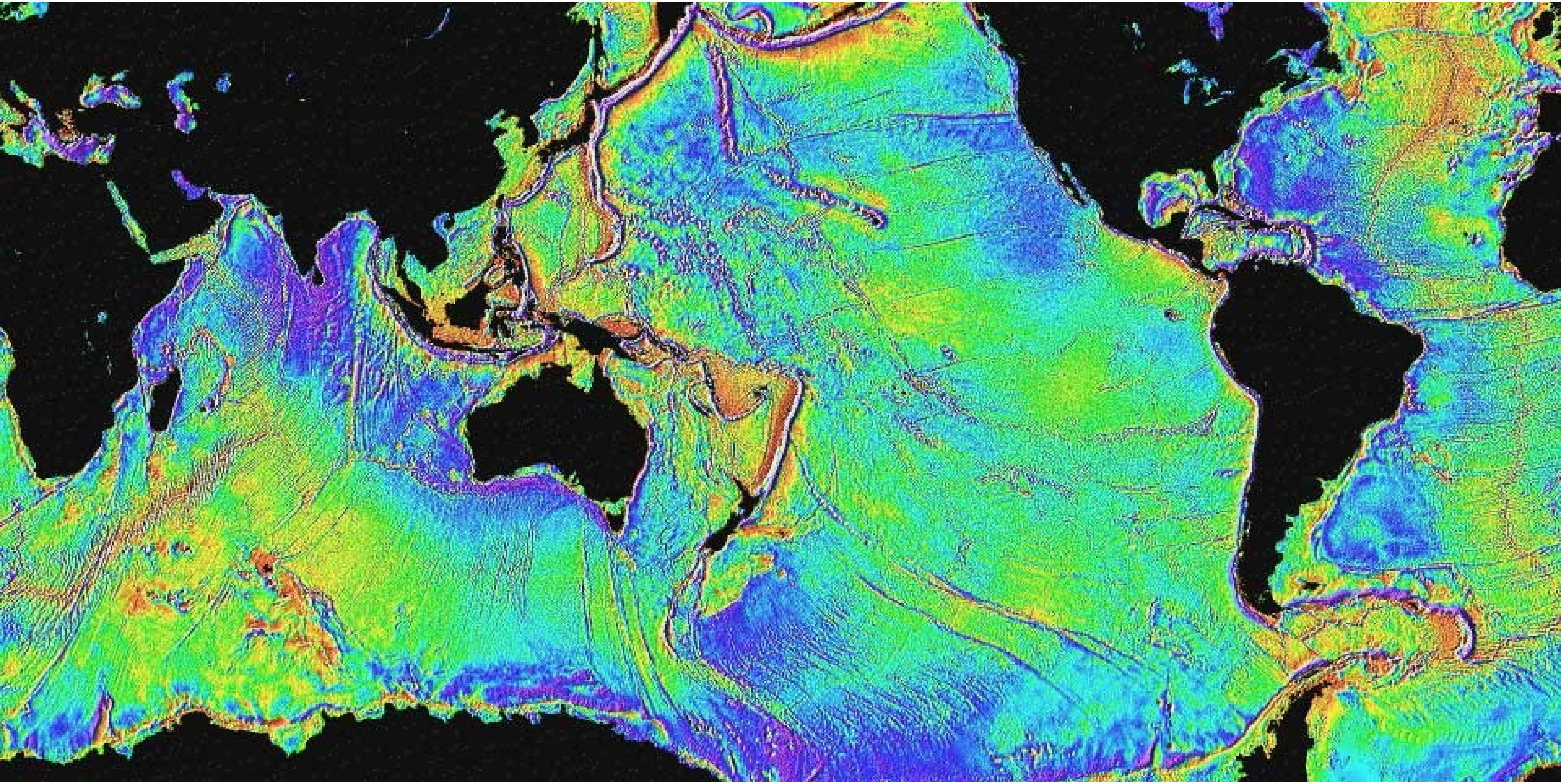
Gravity:

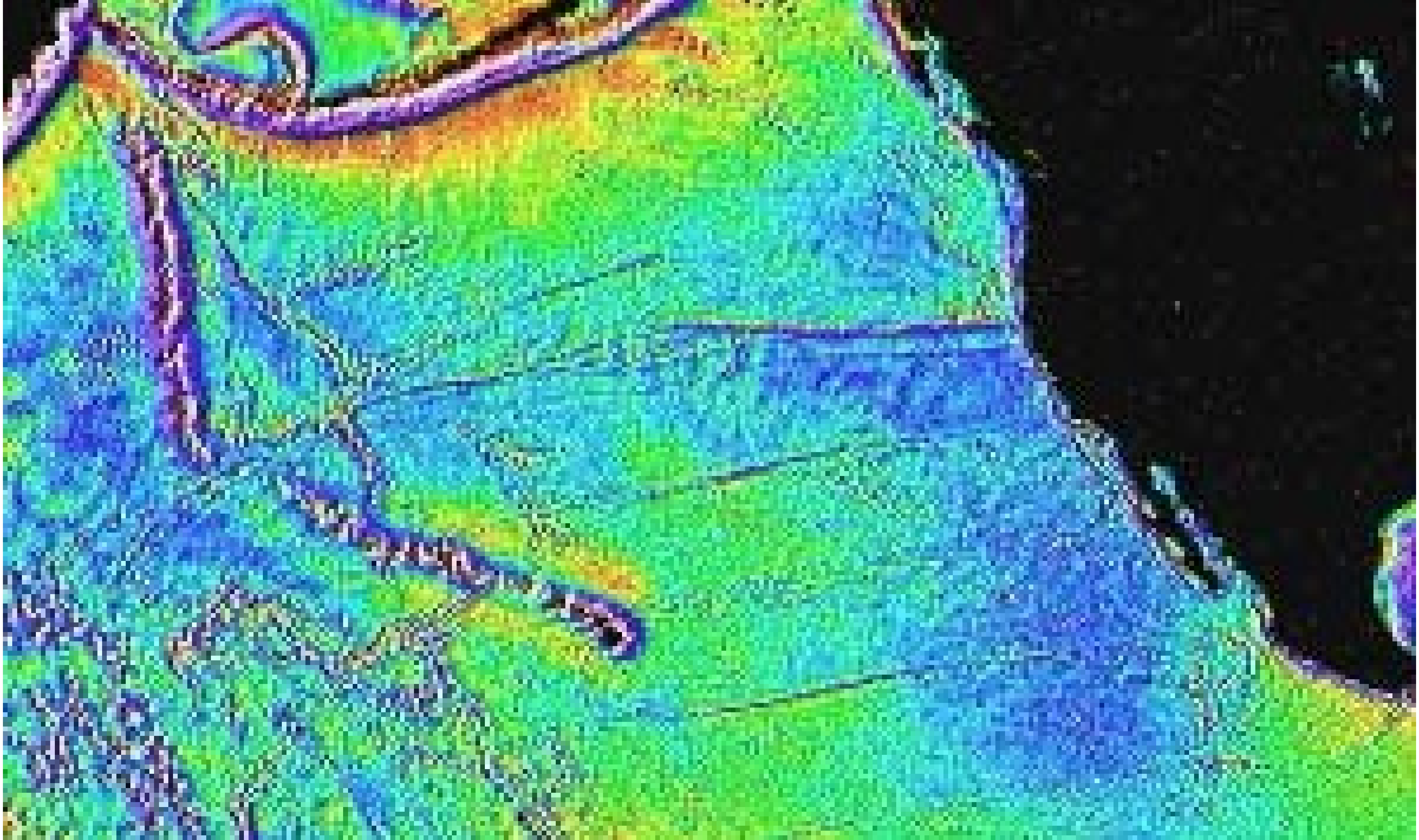
Implications of rotation of the Earth

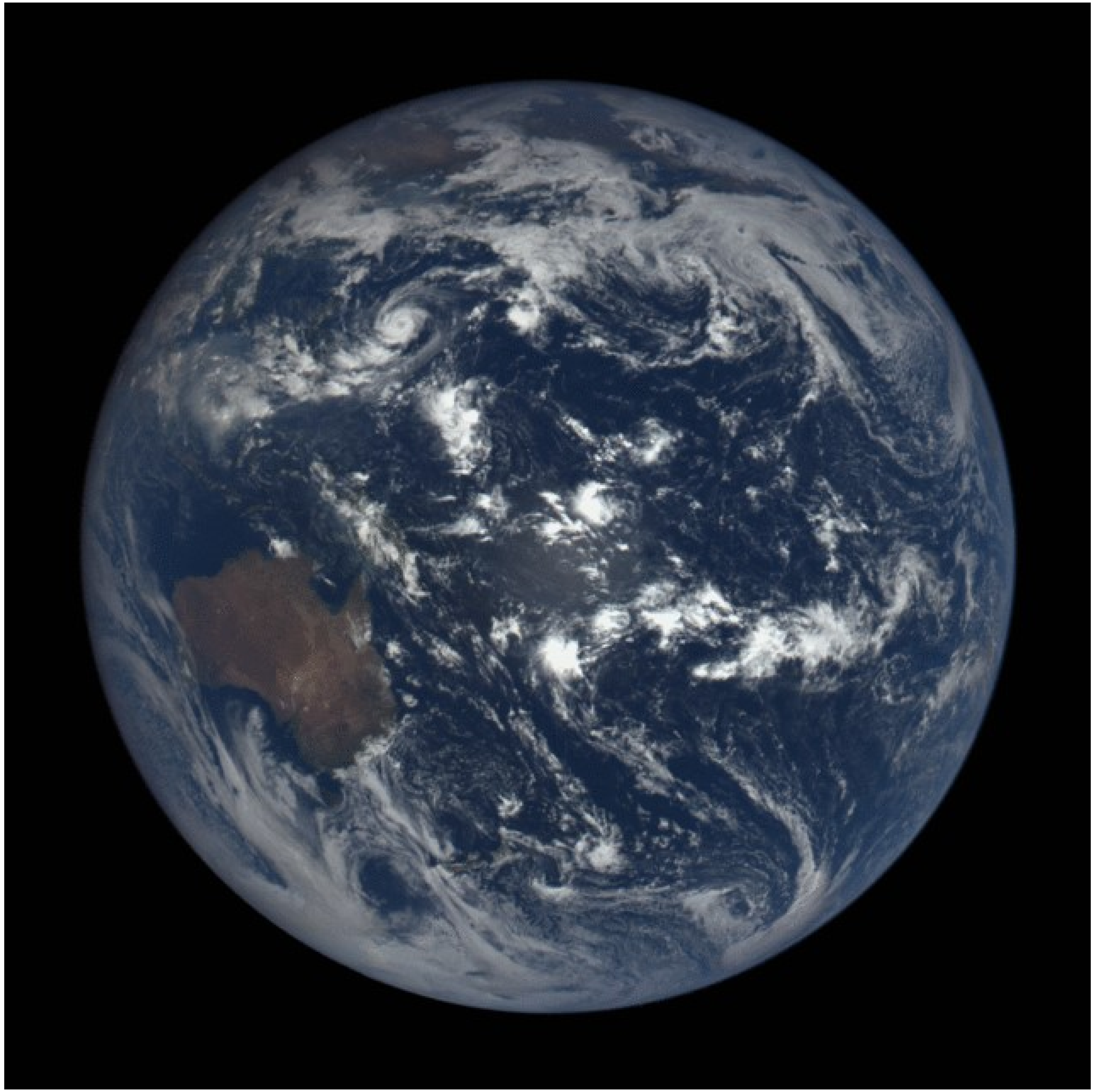
Shape of the Earth


bulges up
sea
surface
Earth

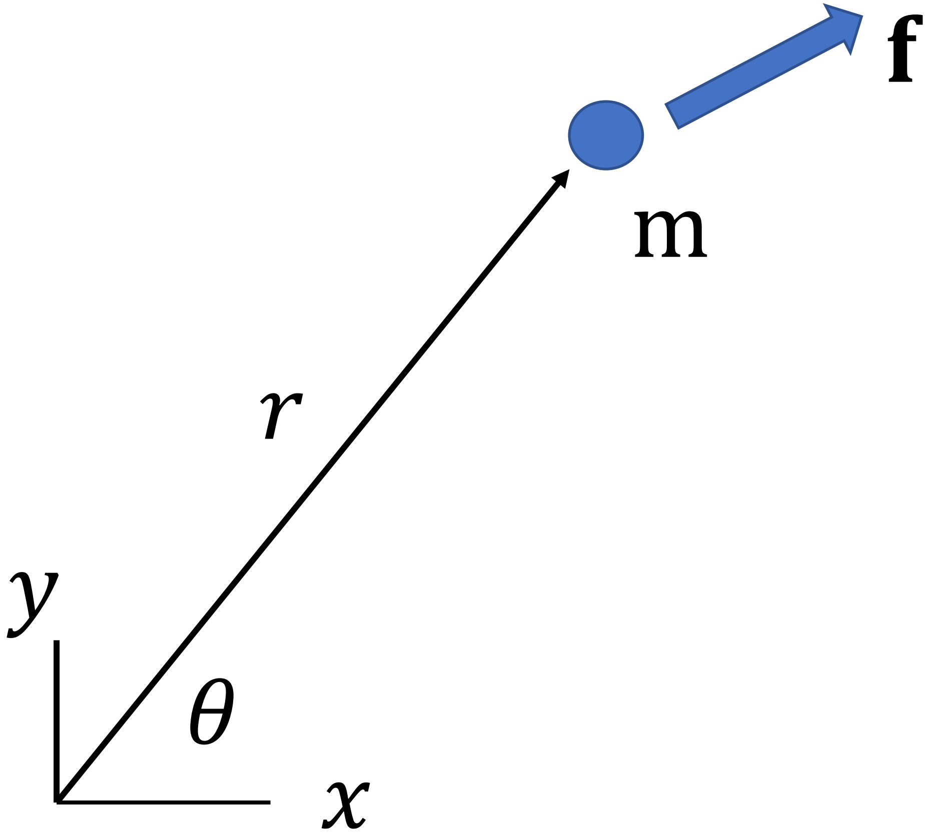

Satellite Radar
Altimetry







English translation
"**And yet it moves**"

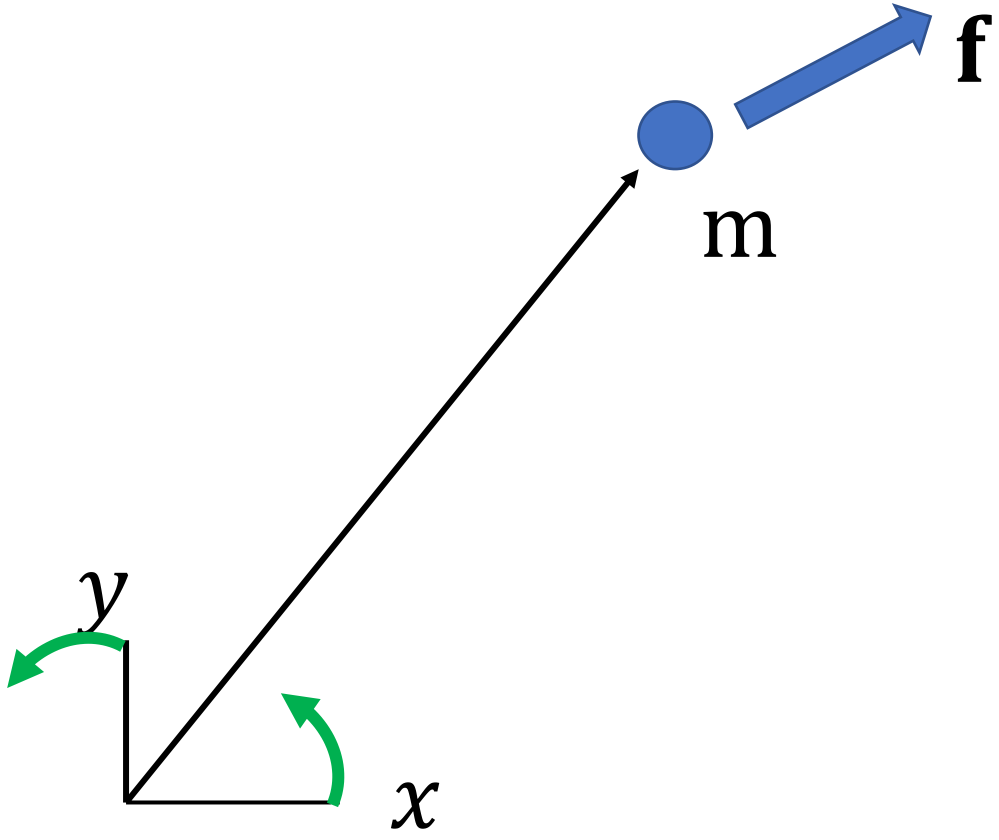


force = mass x acceleration

$$f_x = m\frac{d^2x}{dt^2}$$

$$f_y = m\frac{d^2y}{dt^2}$$

last lecture: different in polar coordinates



force = mass x acceleration

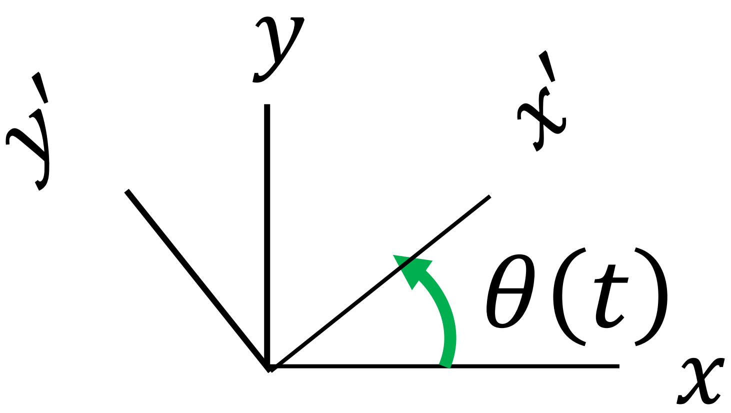
$$f_x = m\frac{d^2x}{dt^2}$$

$$f_y = m\frac{d^2y}{dt^2}$$

TODAY: different in rotating coordinates rotation in plane

$$x' = \quad x \ \cos\theta + y \ \sin\theta$$

$$y' = -x \ \sin\theta + y \ \cos\theta$$

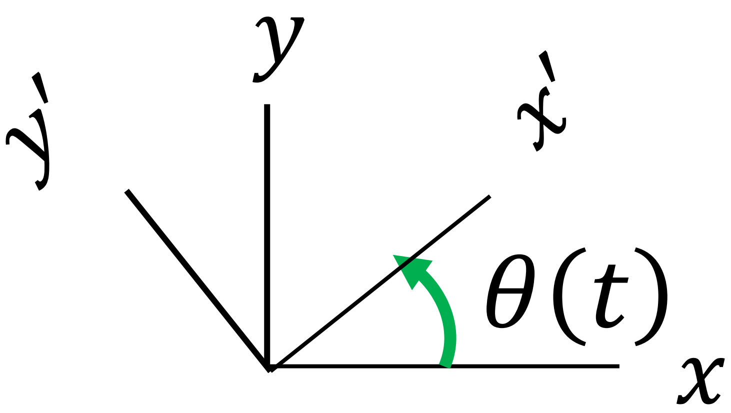


what's $\frac{d^2x}{dt^2}$

what's $\frac{dx}{dt}$

$$x' = \quad x \ \cos\theta + y \ \sin\theta$$

$$y' = -x \ \sin\theta + y \ \cos\theta$$



what's $\frac{d^2x}{dt^2}$

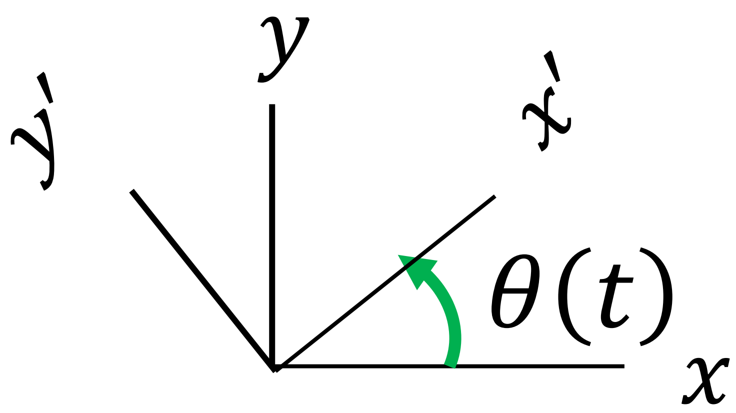
what's $\frac{dx}{dt}$

chain rule

$$\frac{dx\prime}{dt} = \frac{d}{dt}(x \ \cos\theta + y \ \sin\theta) = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta - x \ \sin\theta \frac{d\theta}{dt} + y \ \cos\theta \frac{d\theta}{dt}$$

$$x' = x\ \cos\theta + y\ \sin\theta$$

$$y' = -x\ \sin\theta + y\ \cos\theta$$



what's $\dfrac{d^2x}{dt^2}$

what's $\dfrac{dx}{dt}$

chain rule

$$\frac{dx'}{dt} = \frac{d}{dt}(x\ \cos\theta + y\ \sin\theta) = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta - x\ \sin\theta\frac{d\theta}{dt} + y\ \cos\theta\frac{d\theta}{dt}$$

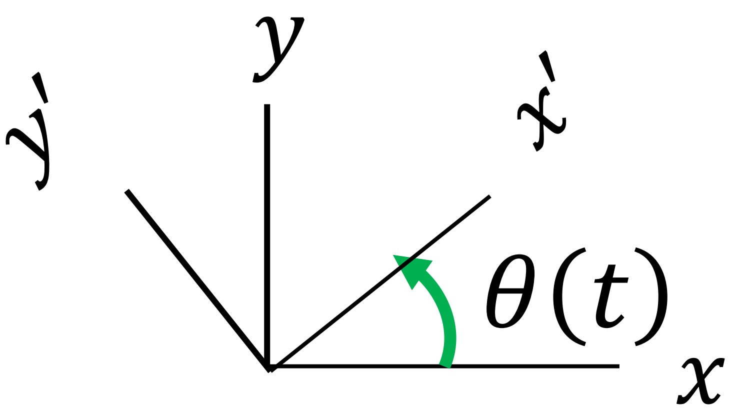
$$= v' \qquad + (-x\sin\theta\ + y\ \cos\theta)\frac{d\theta}{dt}$$

normal velocity

correction term involving spin rate

$$x' = \quad x \ \cos\theta + y \ \sin\theta$$

$$y' = -x \ \sin\theta + y \ \cos\theta$$



what's $\frac{d^2x}{dt^2}$

MESSY!

what's $\frac{dx}{dt}$

chain rule

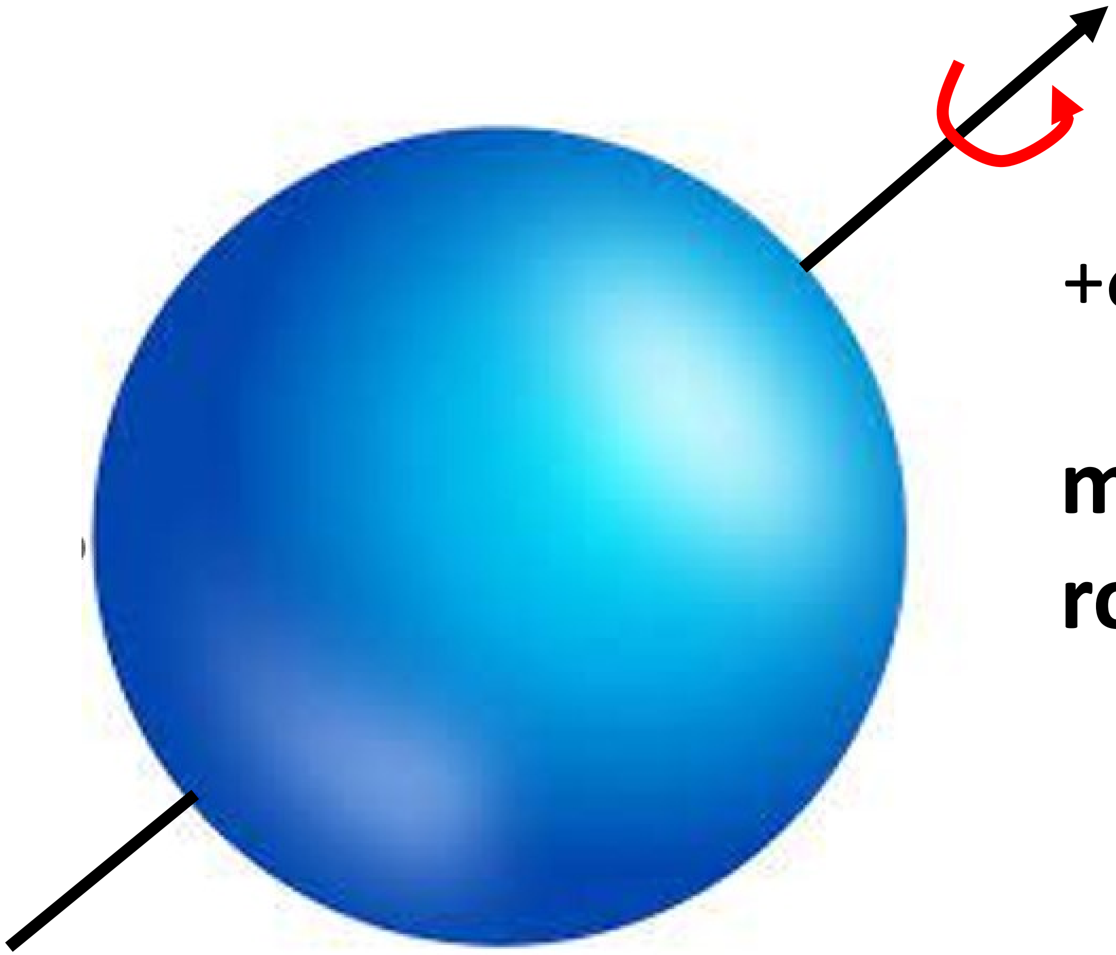
$$\frac{dx'}{dt} = \frac{d}{dt}(x \ \cos\theta + y \ \sin\theta) = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta - x \ \sin\theta\frac{d\theta}{dt} + y \ \cos\theta\frac{d\theta}{dt}$$

$$= v \qquad + \ (-x\sin\theta \ + y \ \cos\theta)\frac{d\theta}{dt}$$

normal velocity

correction term involving spin rate

rotation vector, **ω**



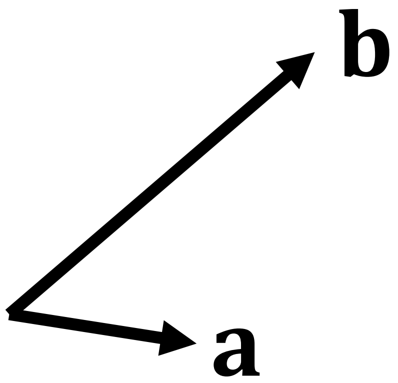
+ω for right hand rule

magnitude, |ω| : rate of rotation in radians per unit time

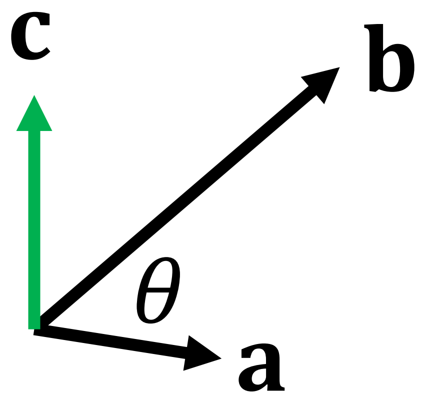
period $T = \frac{2\pi}{|\omega|}$

direction: direction of spin axis

cross product of
two vectors **a** and **b**



cross product of
two vectors $\mathbf{a}$ and $\mathbf{b}$

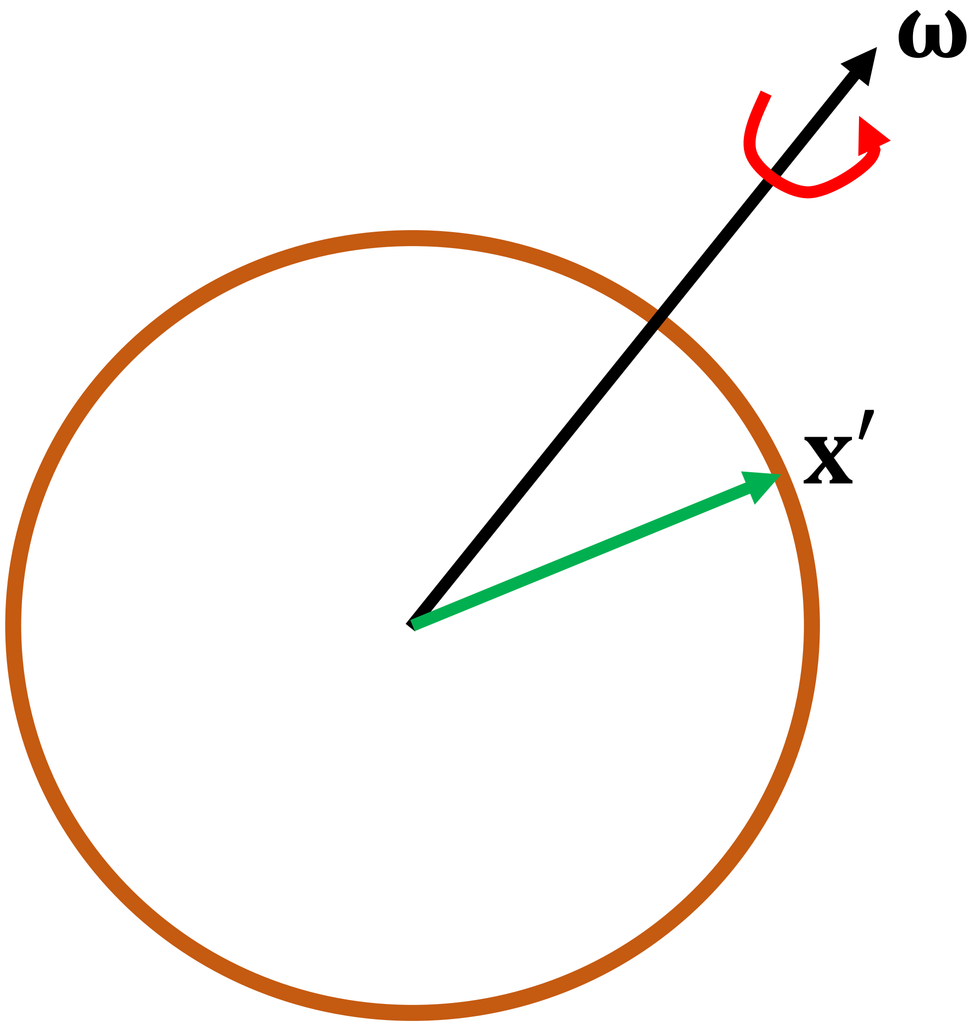


is a vector $\mathbf{c}$

$\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$

direction given by right hand rule of rotating $\mathbf{a}$ towards $\mathbf{b}$

length of $\mathbf{c}$ is length of $\mathbf{a}$ times length of $\mathbf{b}$
times sine of angle between them $|\mathbf{c}| = |\mathbf{a}||\mathbf{b}| \sin\theta$

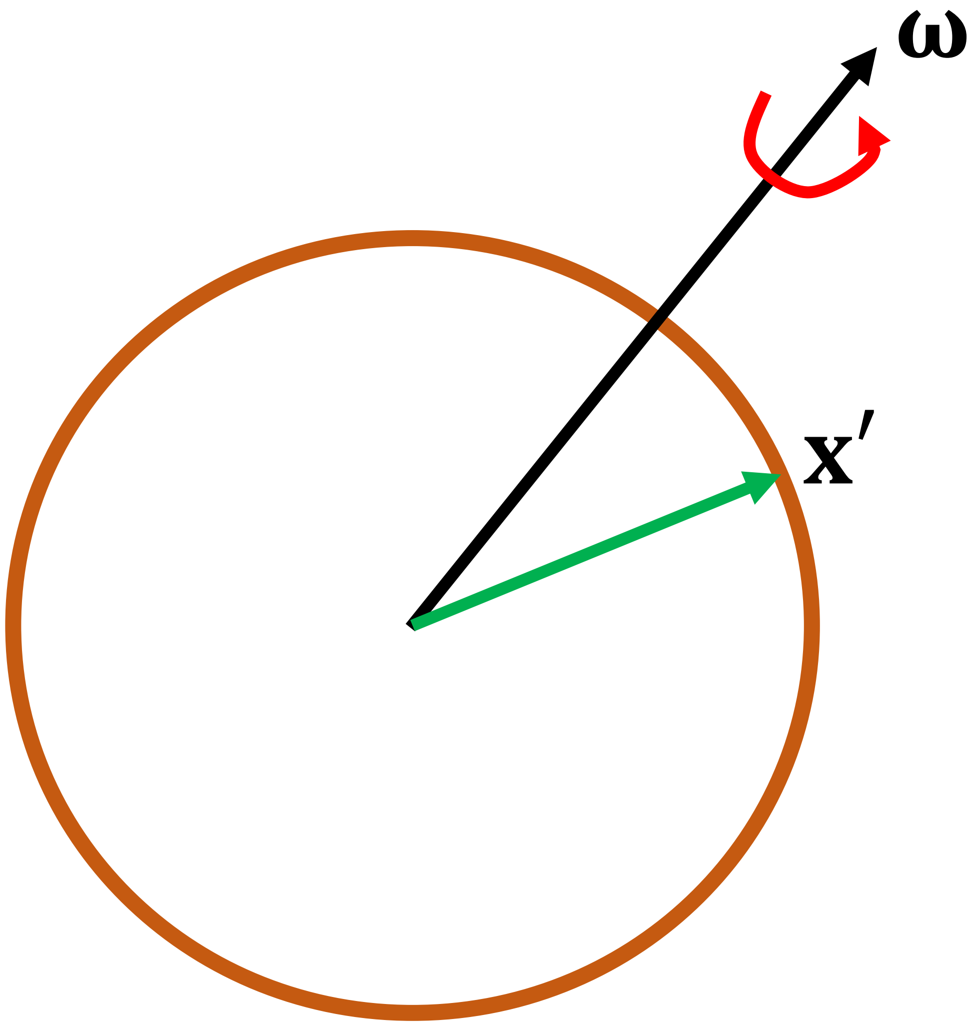


$$\frac{d^2\mathbf{x}}{dt^2} = \frac{d^2\mathbf{x}'}{dt^2}$$

$$+m\boldsymbol{\omega}' \times (\boldsymbol{\omega}' \times \mathbf{x}')$$

$$+\,2m\boldsymbol{\omega}' \times \mathbf{v}'$$

$$+\,\frac{d\boldsymbol{\omega}'}{dt} \times \mathbf{x}'$$

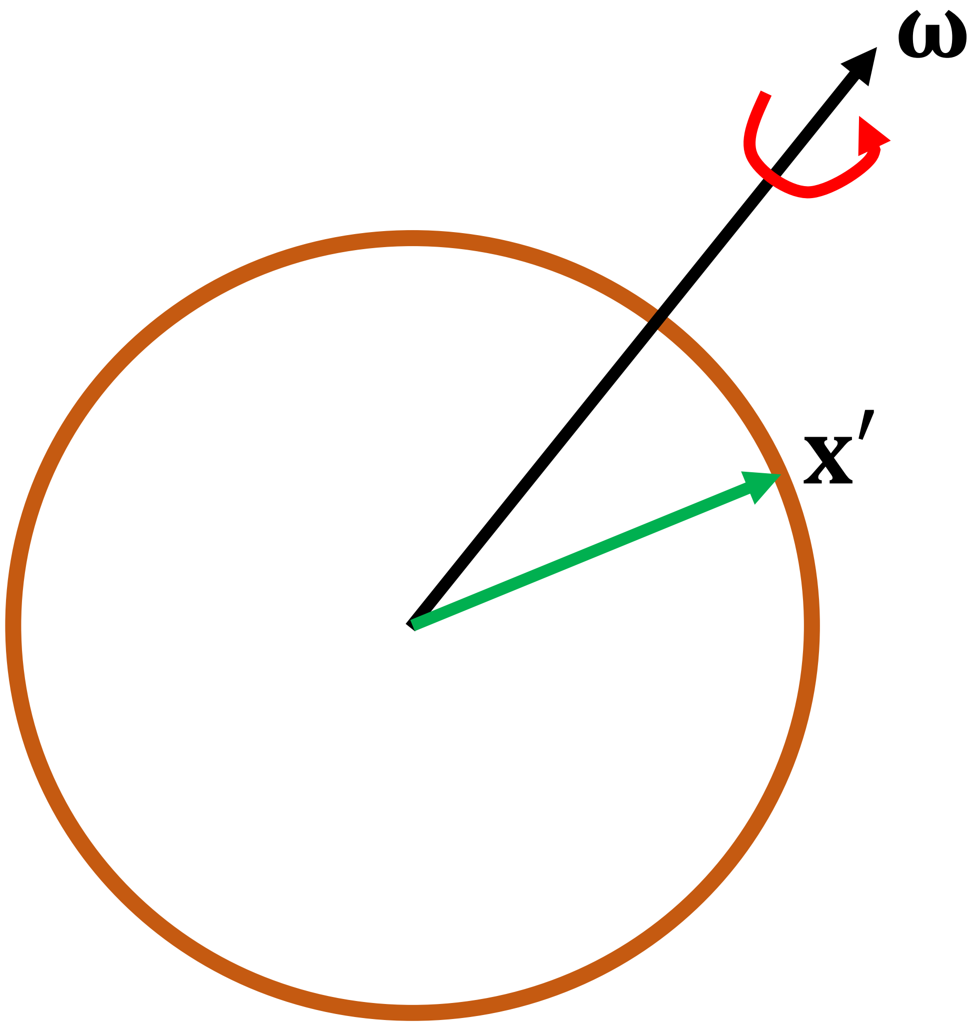


$$\mathbf{f} = m\mathbf{a}'$$

$$+m\boldsymbol{\omega}' \times (\boldsymbol{\omega}' \times \mathbf{x}')$$

$$+\,2m\boldsymbol{\omega}' \times \mathbf{v}'$$

$$+\,\frac{d\boldsymbol{\omega}'}{dt} \times \mathbf{x}'$$

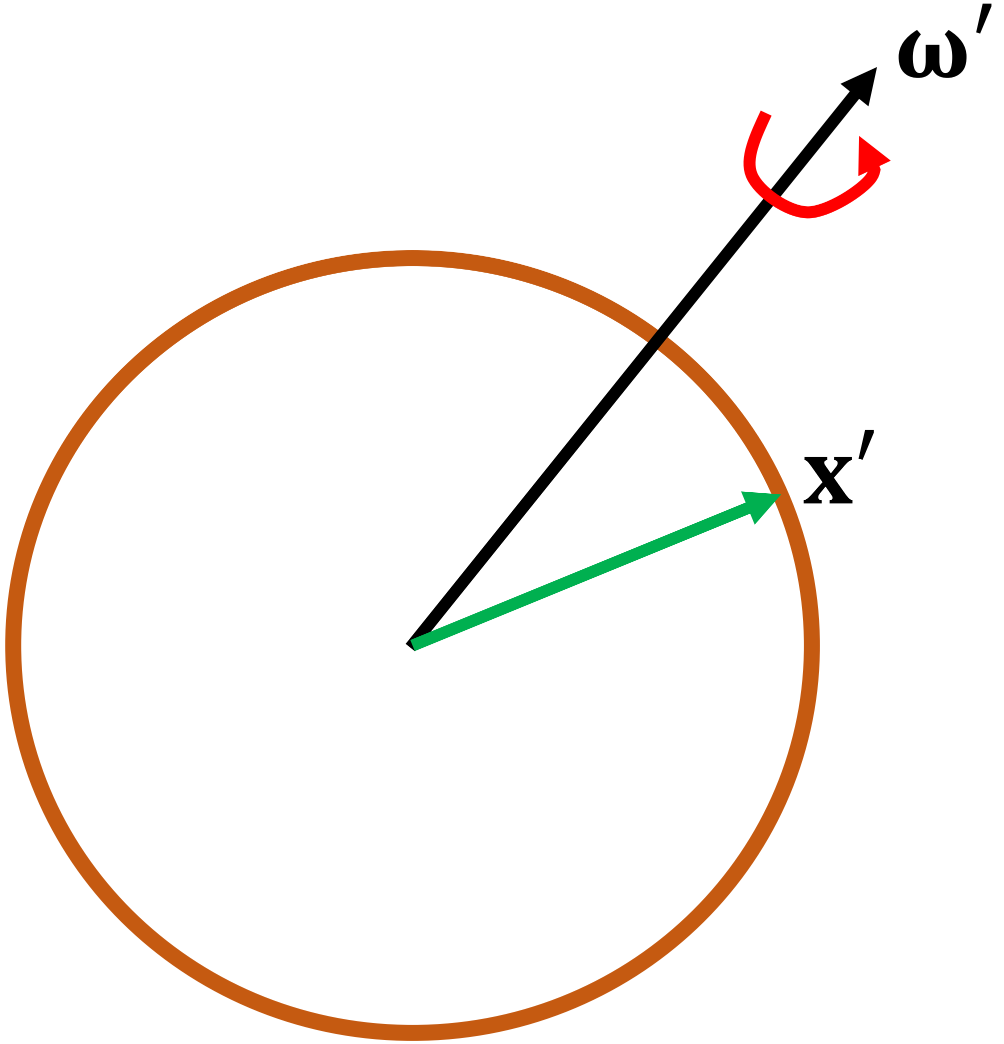


$$m\mathbf{a}' = \mathbf{f}' +$$

$$-m\boldsymbol{\omega}' \times (\boldsymbol{\omega}' \times \mathbf{x}')$$

$$-2m\boldsymbol{\omega}' \times \mathbf{v}'$$

$$-\frac{d\boldsymbol{\omega}'}{dt} \times \mathbf{x}'$$



$$\mathbf{ma}' = \mathbf{f}' +$$

$$-\mathrm{m}\boldsymbol{\omega}' \times (\boldsymbol{\omega}' \times \mathbf{x}')$$

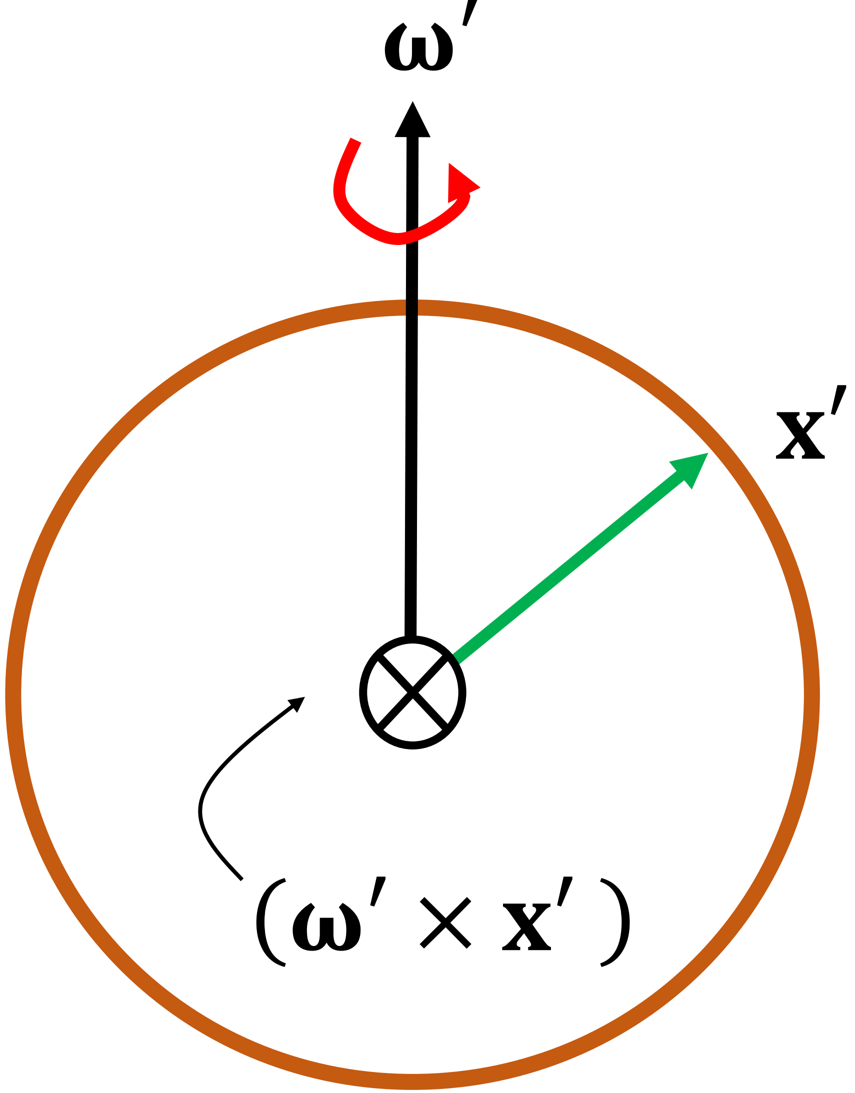
centrifugal force

$$-2\mathrm{m}\boldsymbol{\omega}' \times \mathbf{v}'$$

coriollis force

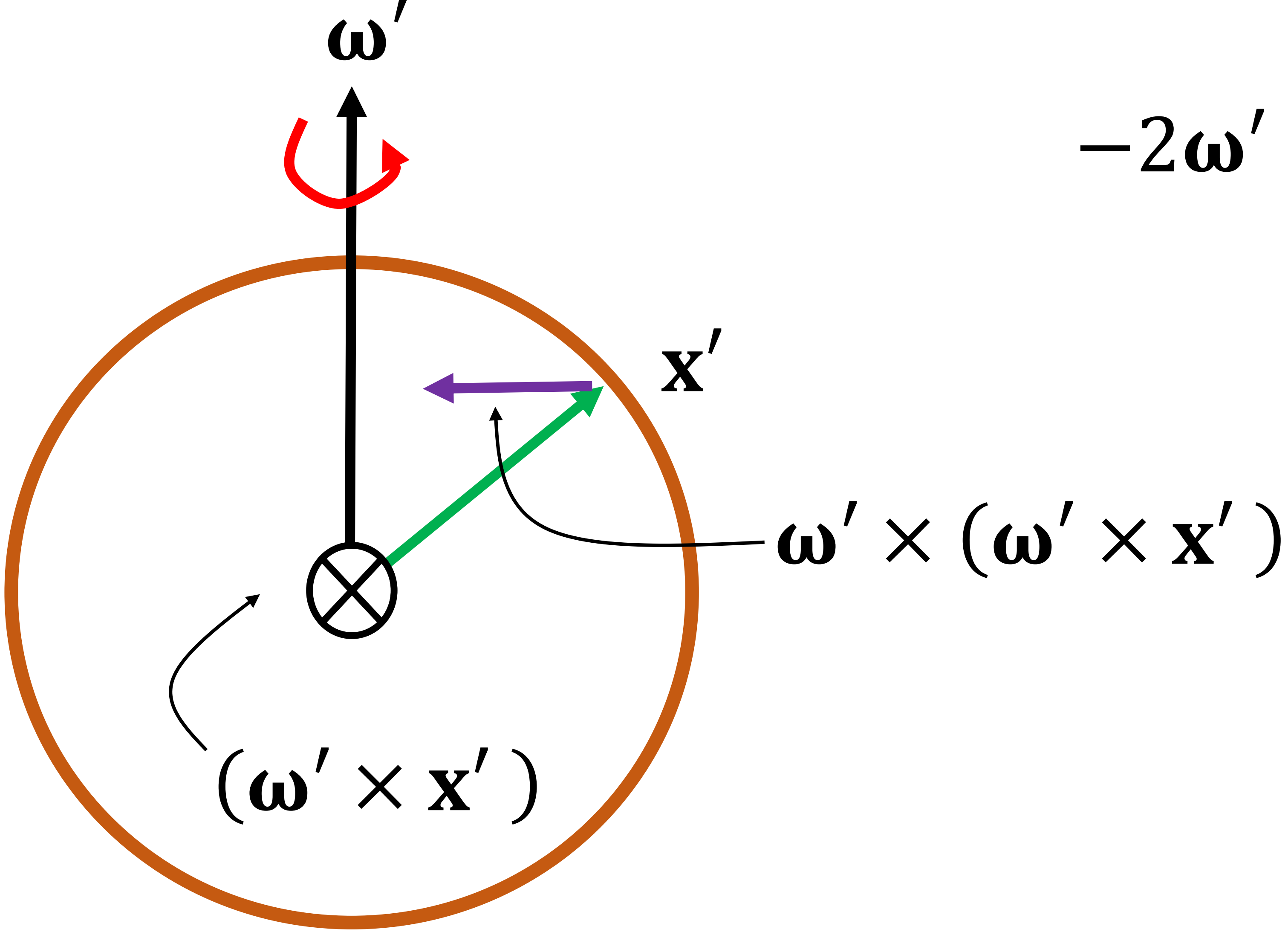
$$-\frac{d\boldsymbol{\omega}'}{dt} \times \mathbf{x}'$$

wobble force



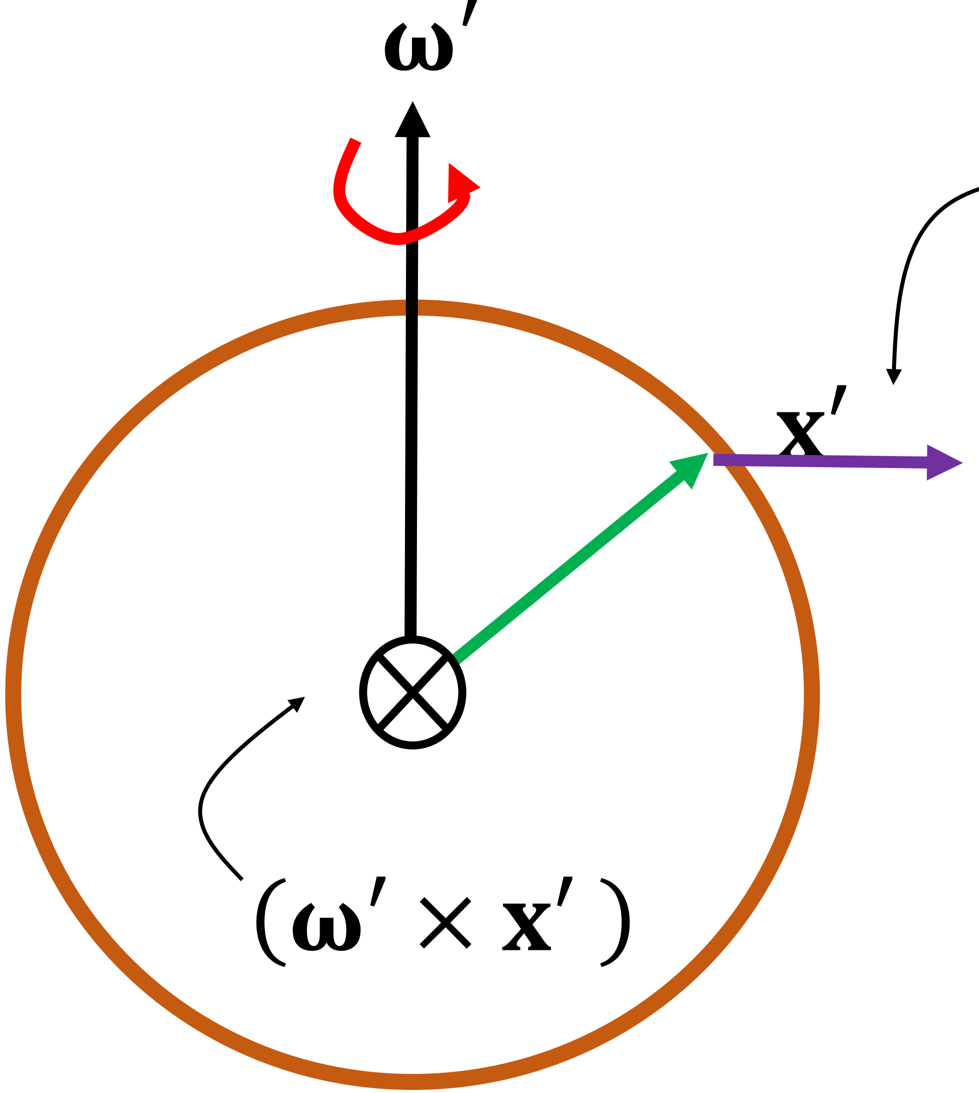
$$-2\boldsymbol{\omega}' \times (\boldsymbol{\omega}' \times \mathbf{x}')$$

centrifugal force



$$-2\boldsymbol{\omega}' \times (\boldsymbol{\omega}' \times \mathbf{x}')$$

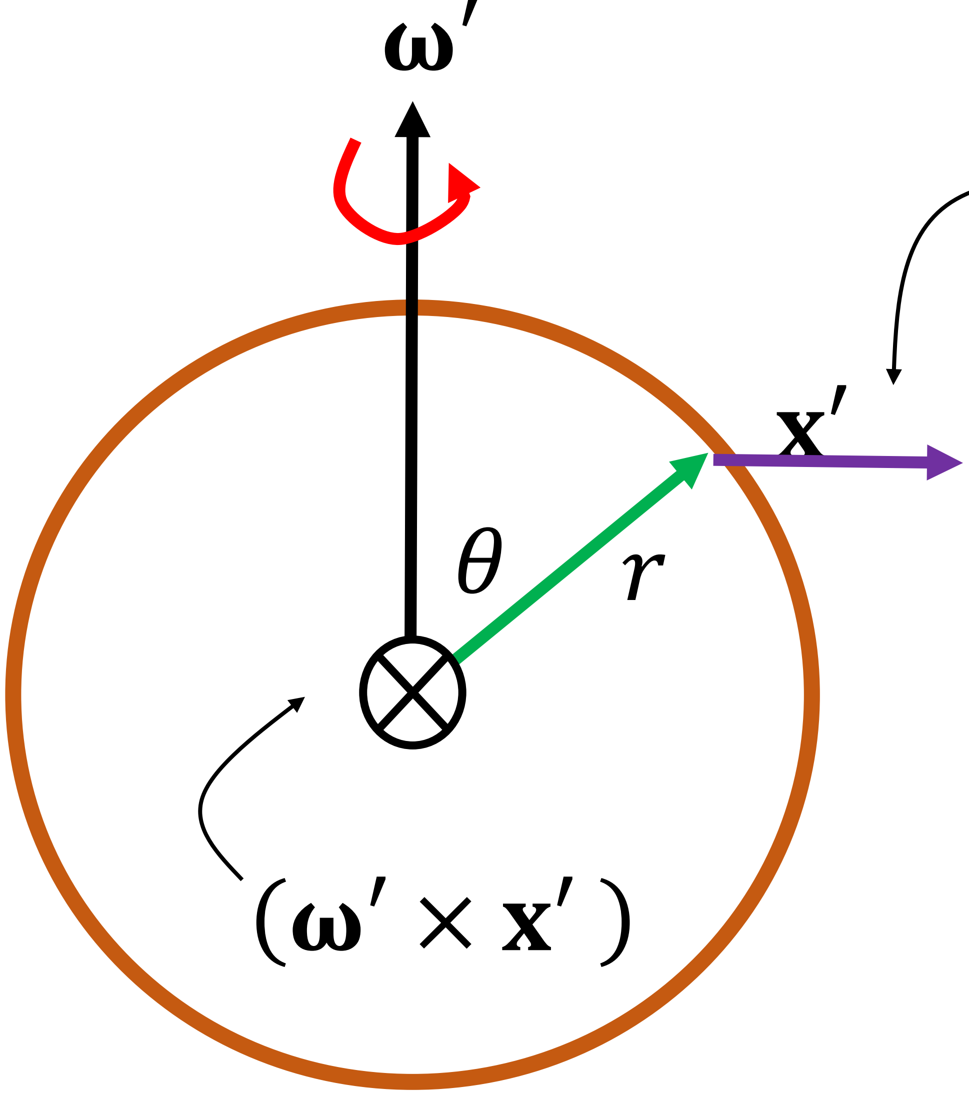
centrifugal force



$$-m\boldsymbol{\omega}' \times (\boldsymbol{\omega}' \times \mathbf{x}')$$

centrifugal force

perpendicular away
from to spin axis



$$-m\boldsymbol{\omega}' \times (\boldsymbol{\omega}' \times \mathbf{x}')$$

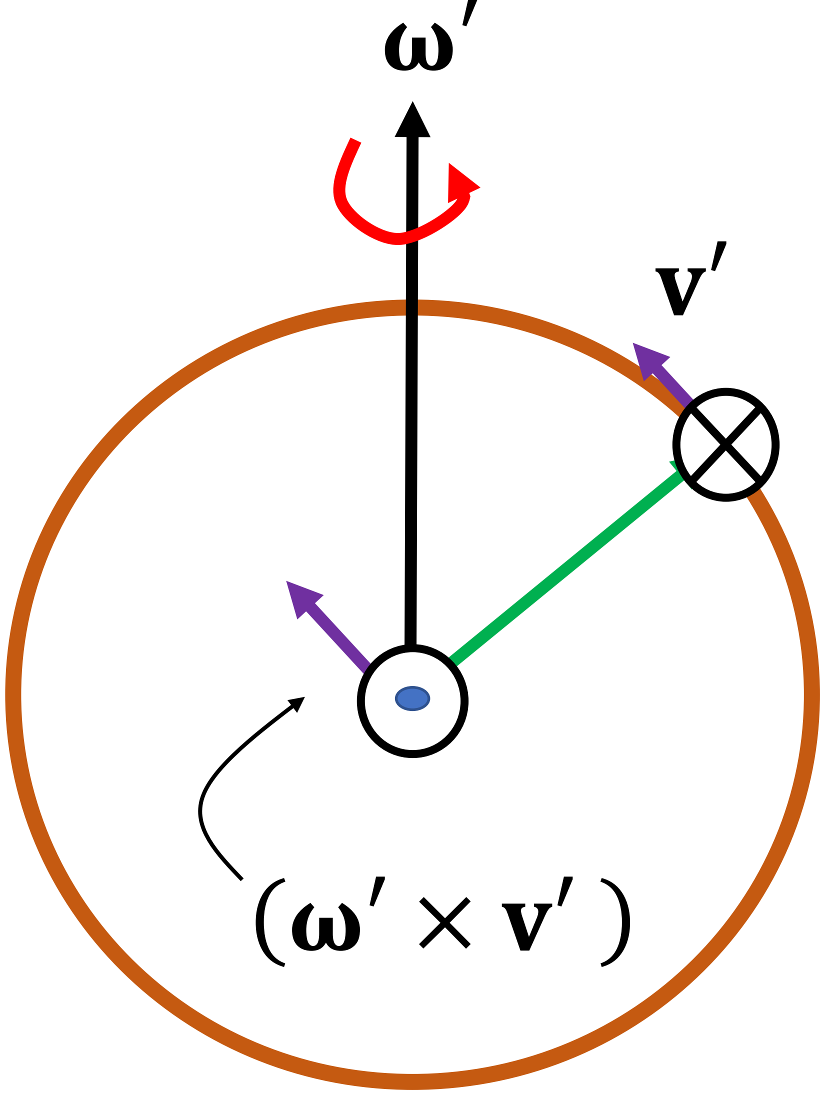
centrifugal force

magnitude

$$m|\boldsymbol{\omega}'|^2 r \sin\theta$$

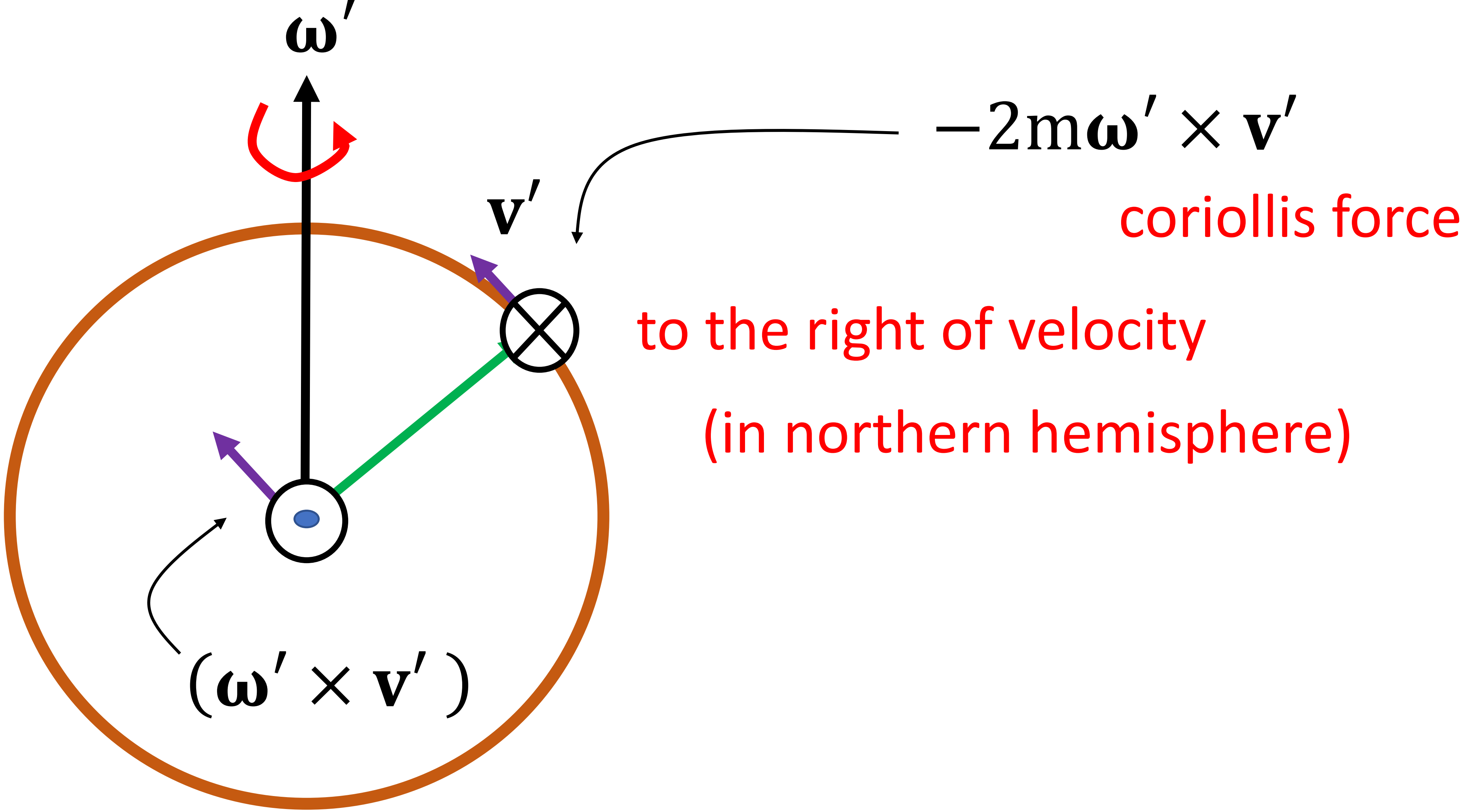
biggest at equator

zero at poles

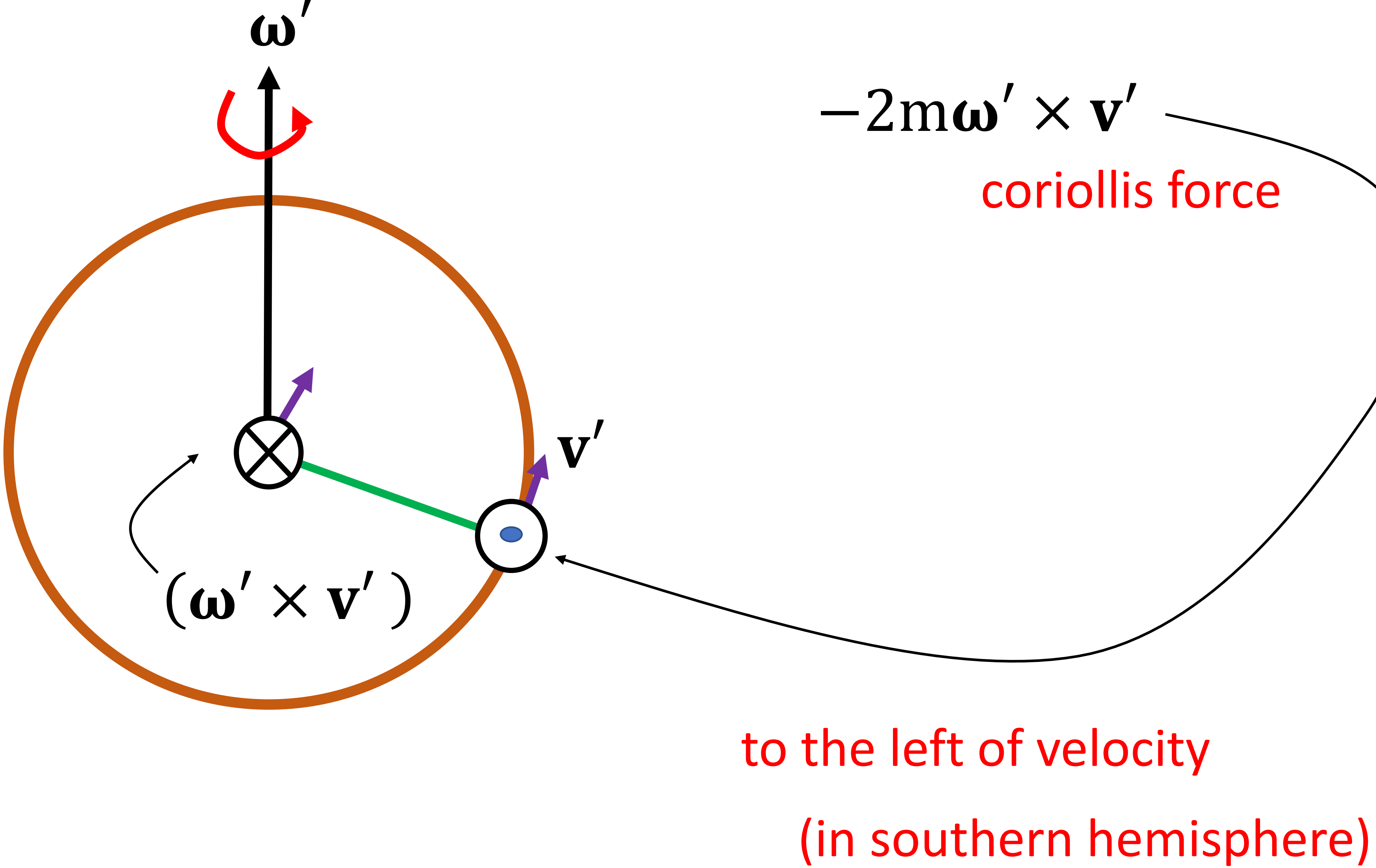


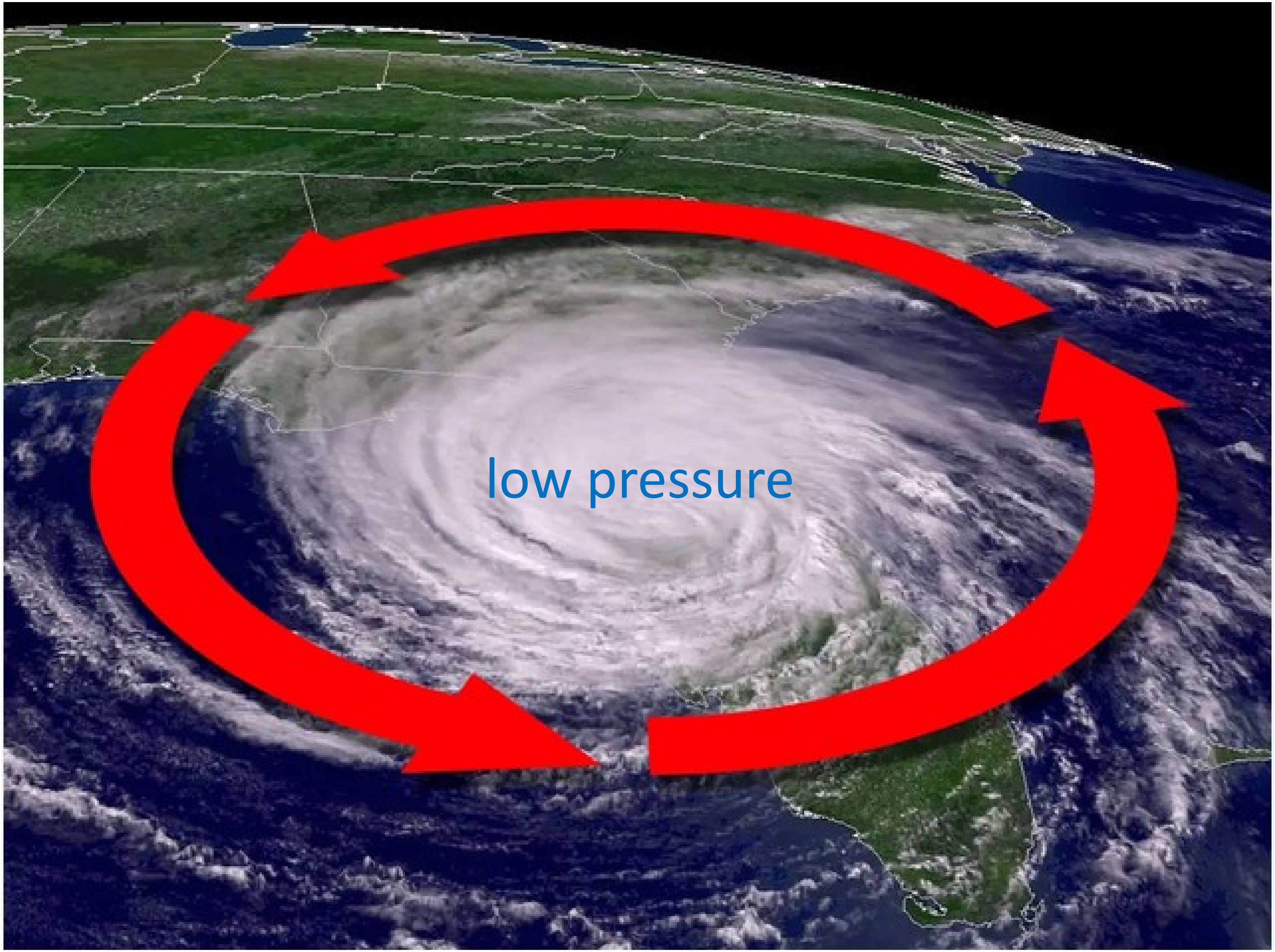
$$-2m\boldsymbol{\omega}' \times \mathbf{v}'$$

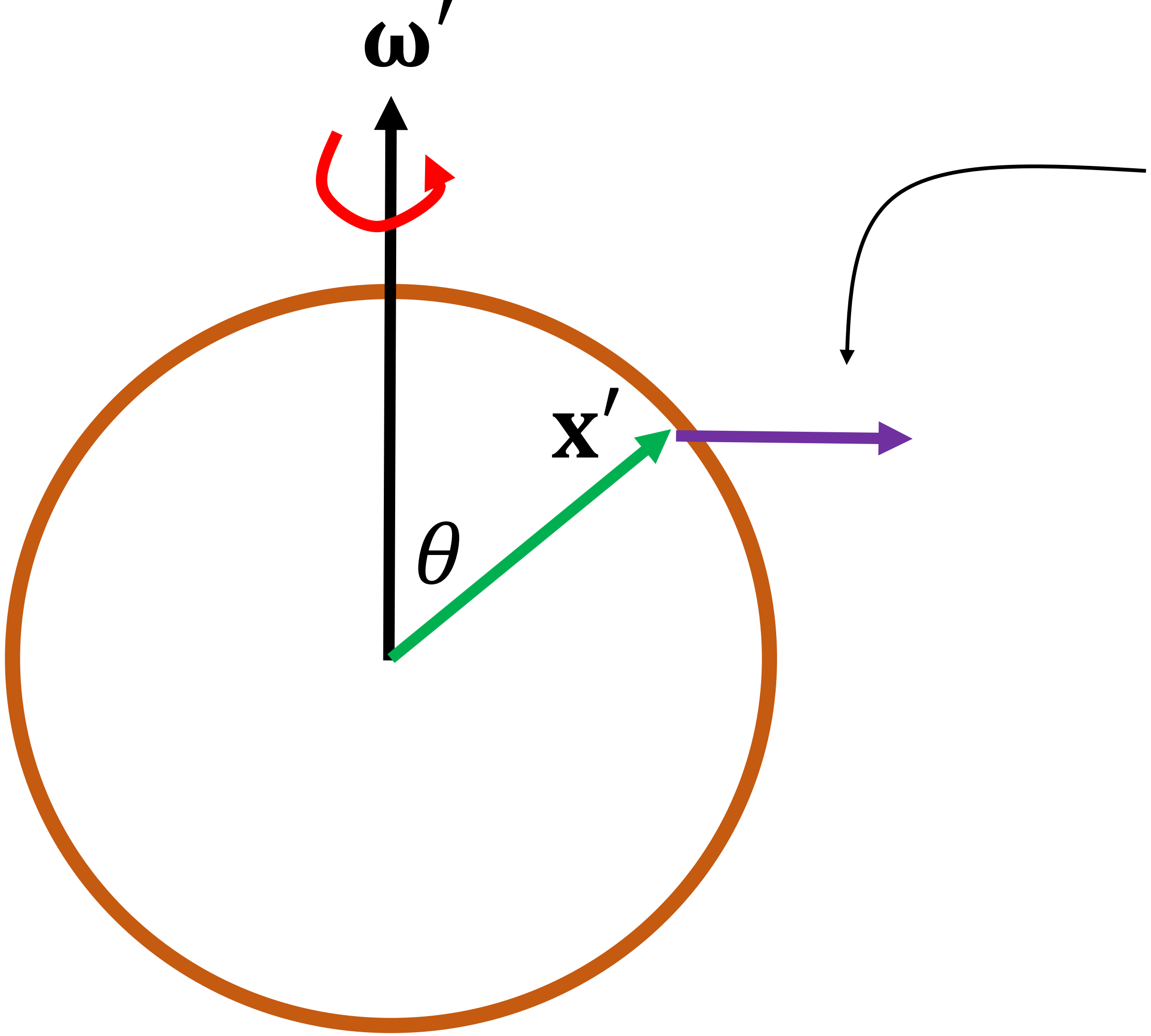
coriollis force



$\boldsymbol{\omega}'$
$-2m\boldsymbol{\omega}' \times \mathbf{v}'$
$\mathbf{v}'$
coriollis force
to the right of velocity
(in northern hemisphere)
$(\boldsymbol{\omega}' \times \mathbf{v}')$


ω′
−2mω′ × v′
coriollis force
v′
(ω′ × v′)
to the left of velocity
(in southern hemisphere)


low pressure



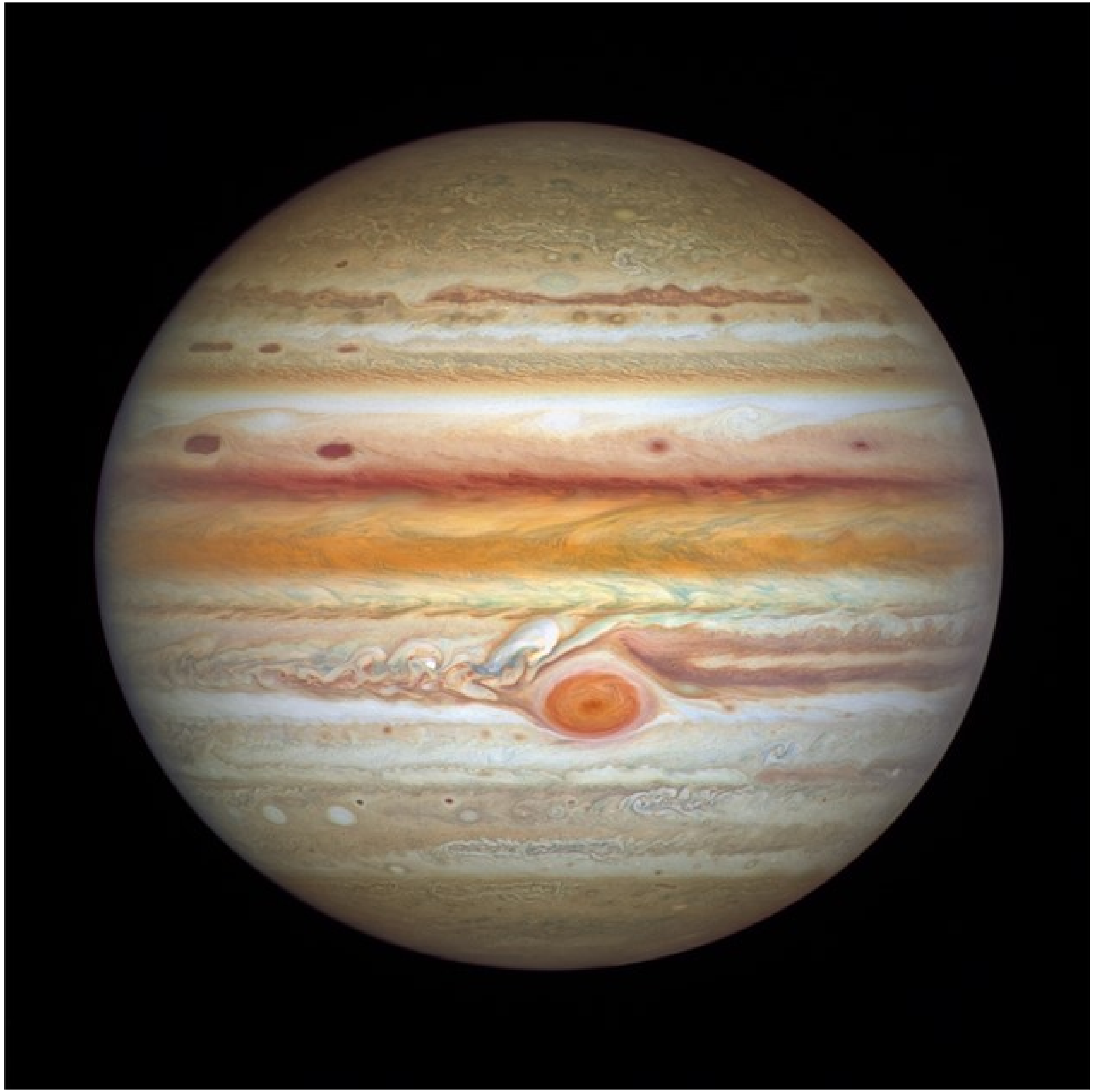
centrifugal force

perpendicular away from to spin axis

magnitude

$$m|\boldsymbol{\omega}'|^2 r \sin\theta$$

The centrifugal force looks like anti-gravity. So how does it affect the shape of the Earth?

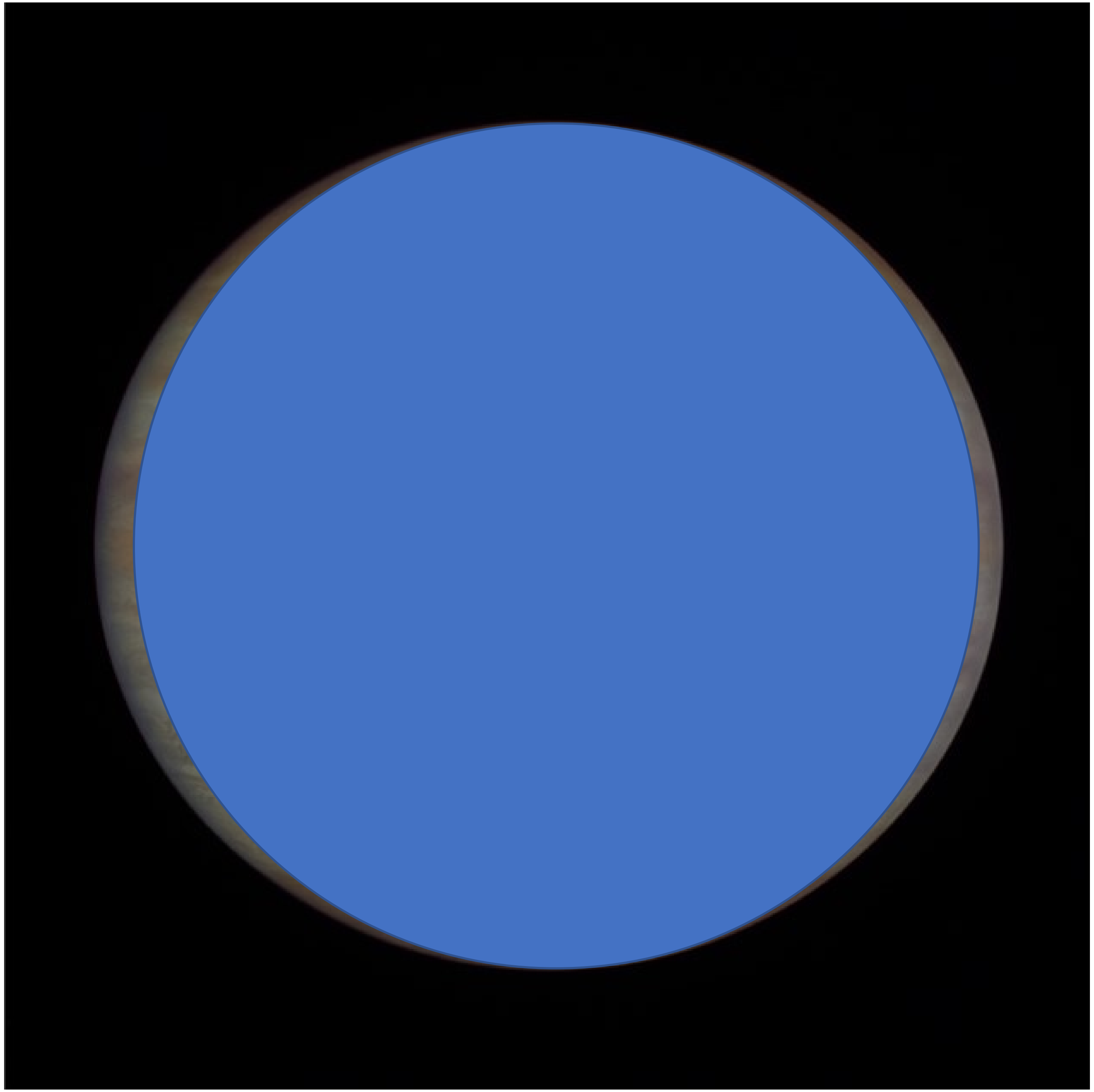


Jupiter

Radius 11.209 of Earth’s

Length of day
9 h 55 m

bigger and
spinning faster



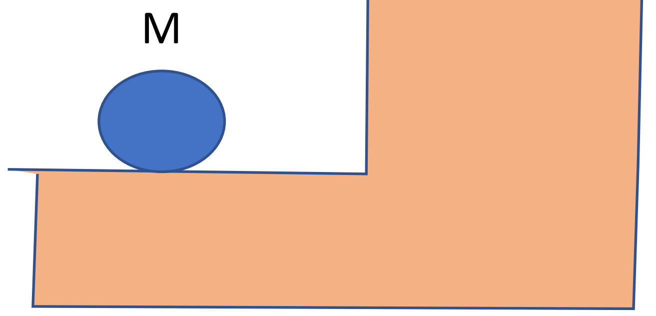
Jupiter

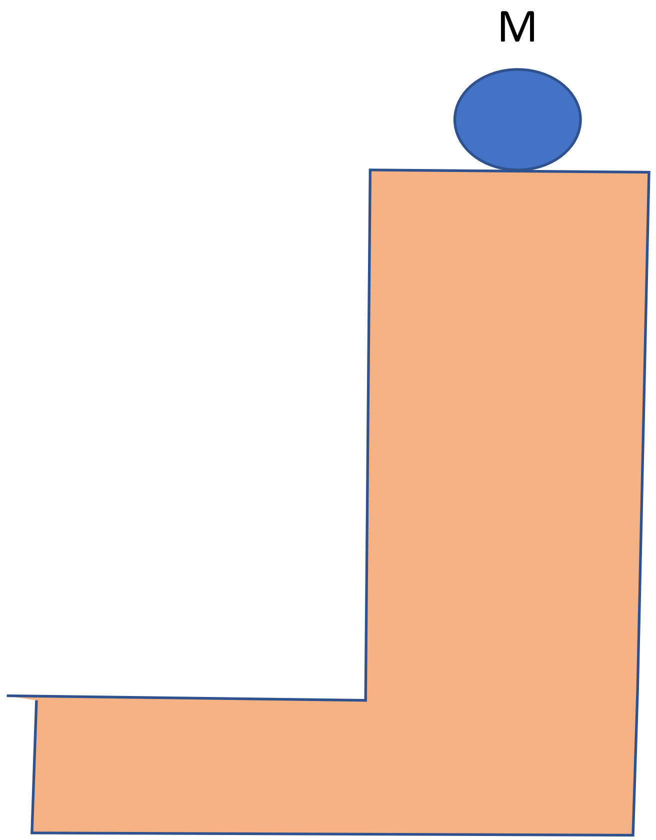
Radius 11.209 of Earth’s

Length of day
9 h 55 m

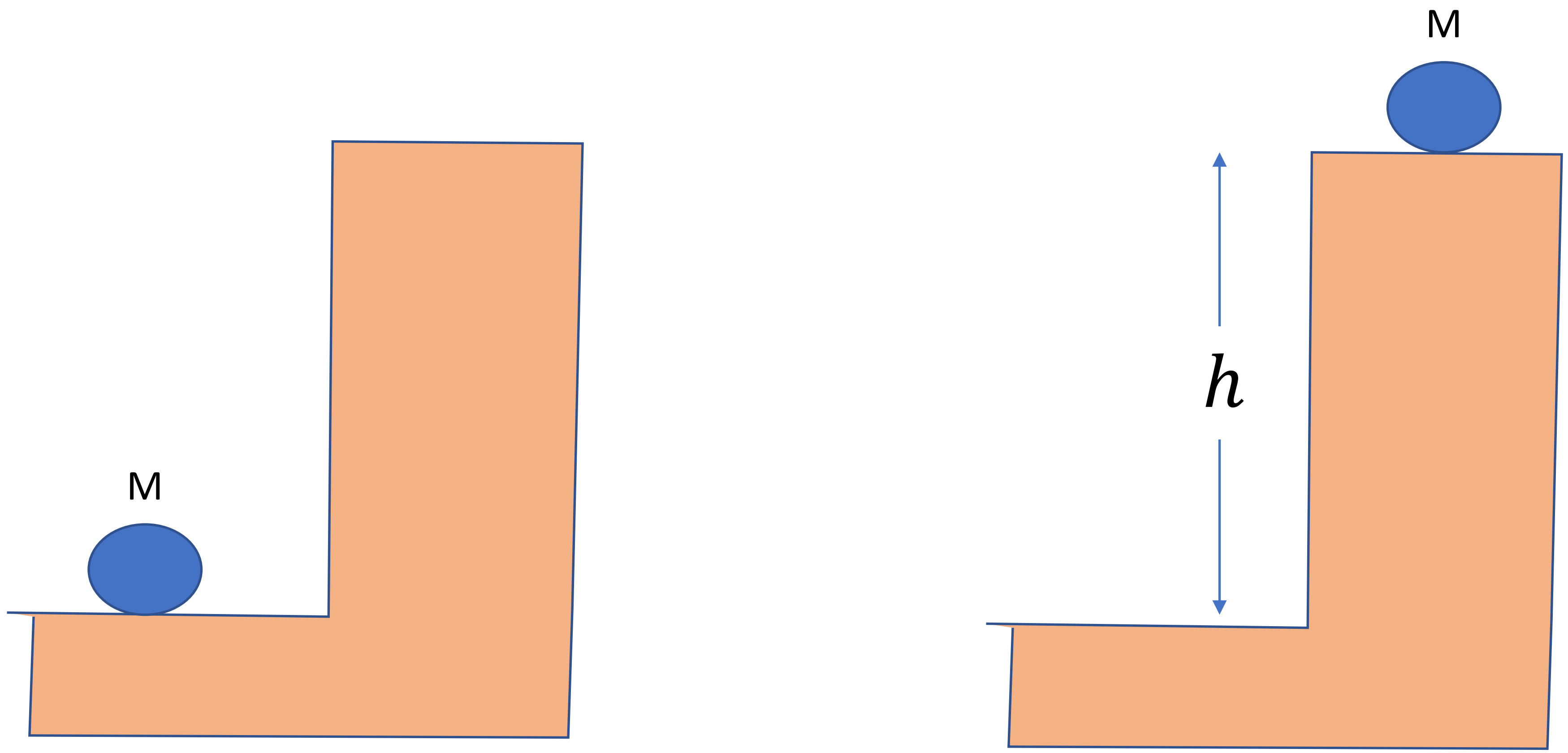
bigger and spinning faster

Energy expenditure: moving against a force



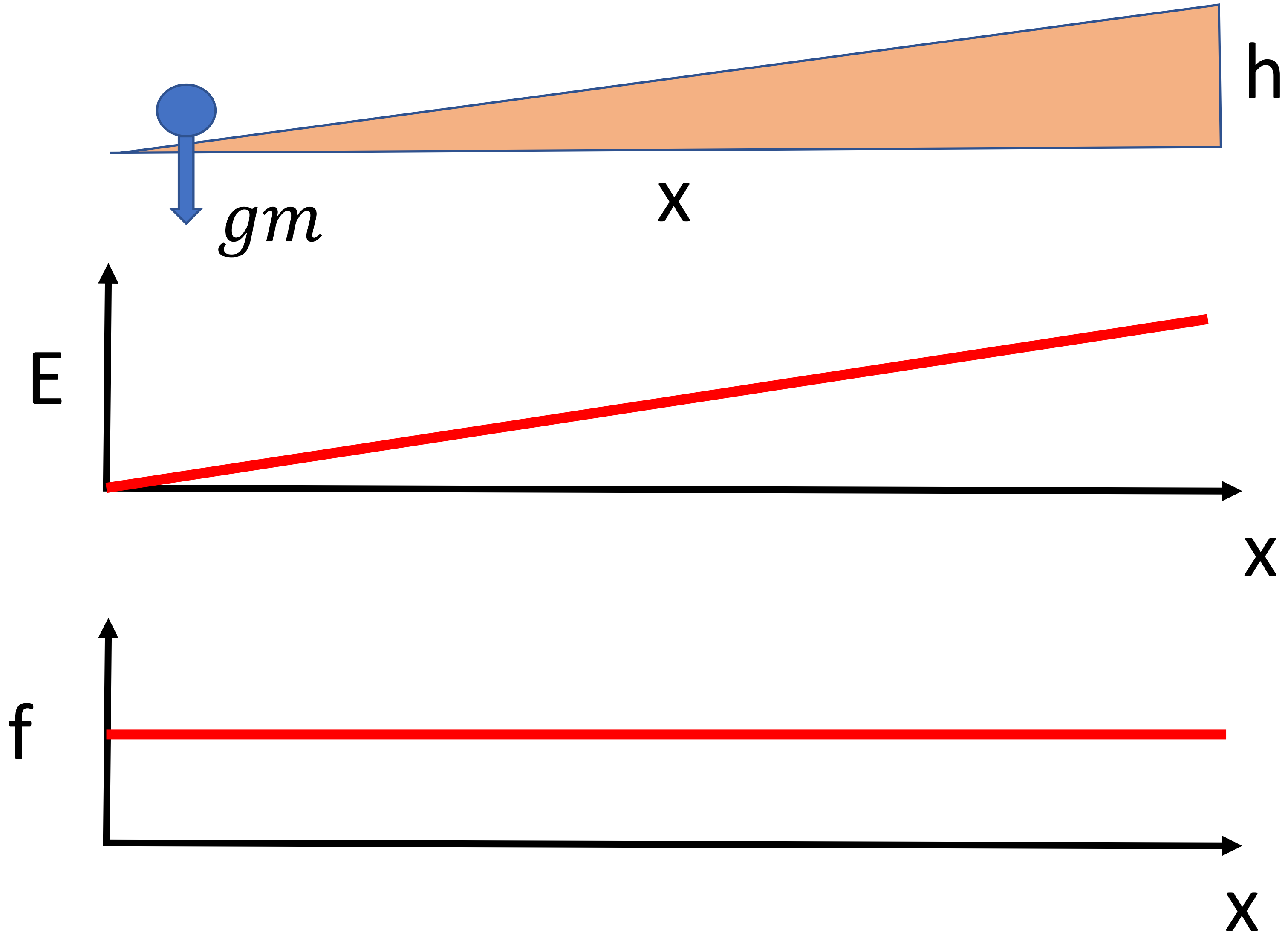


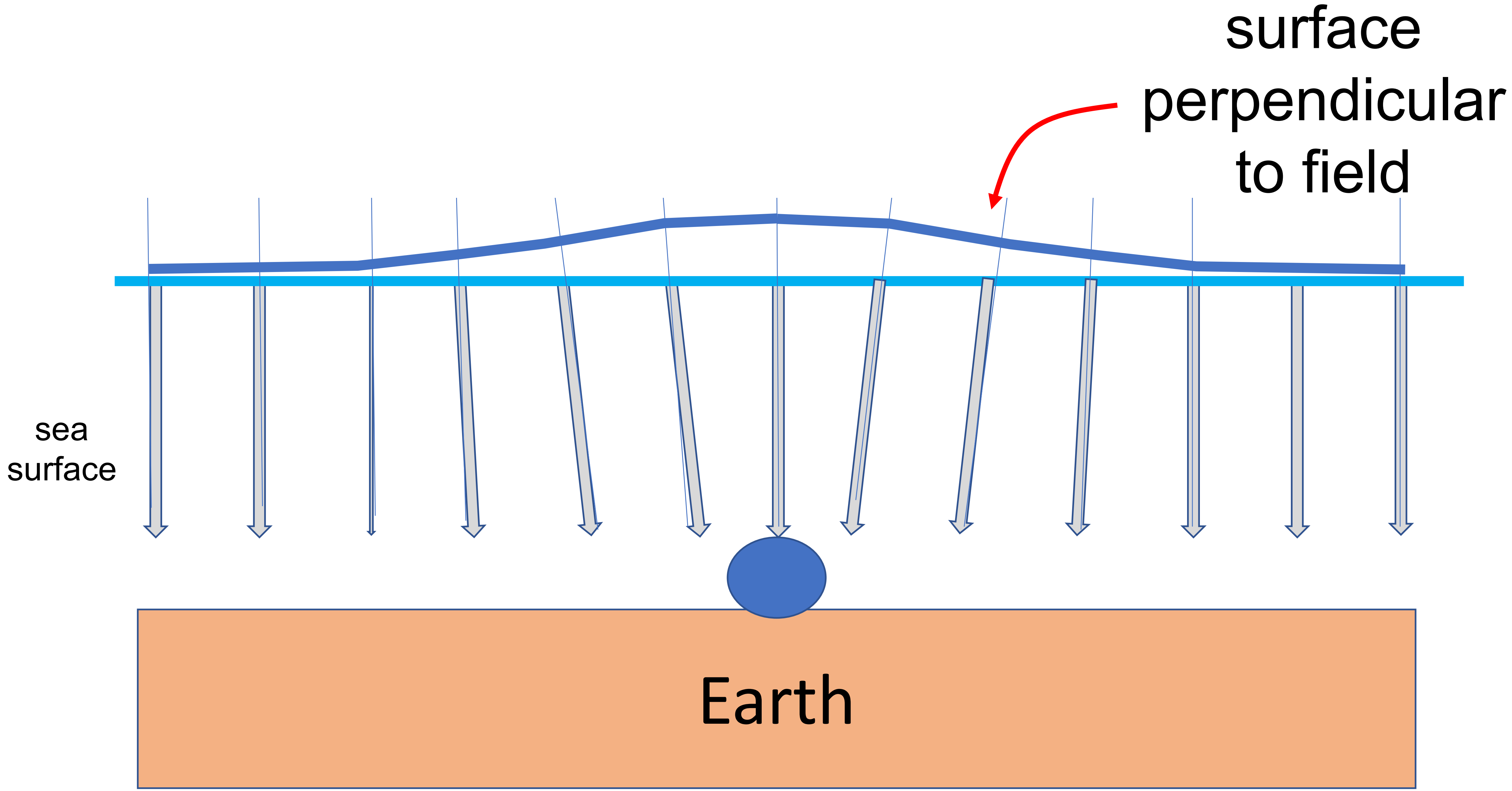
Energy expenditure: moving against a force

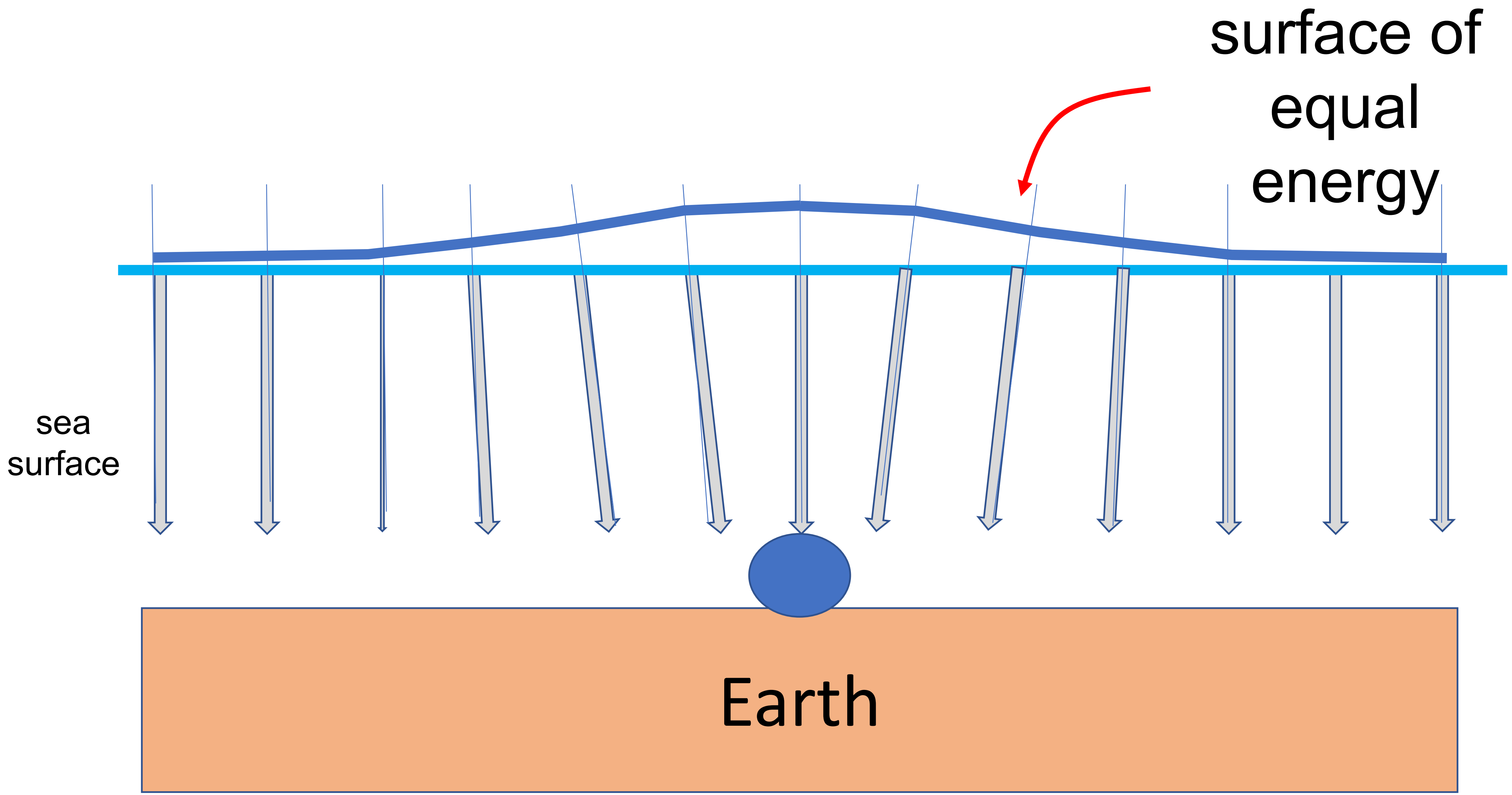


E = force × distance = $gm \times h$

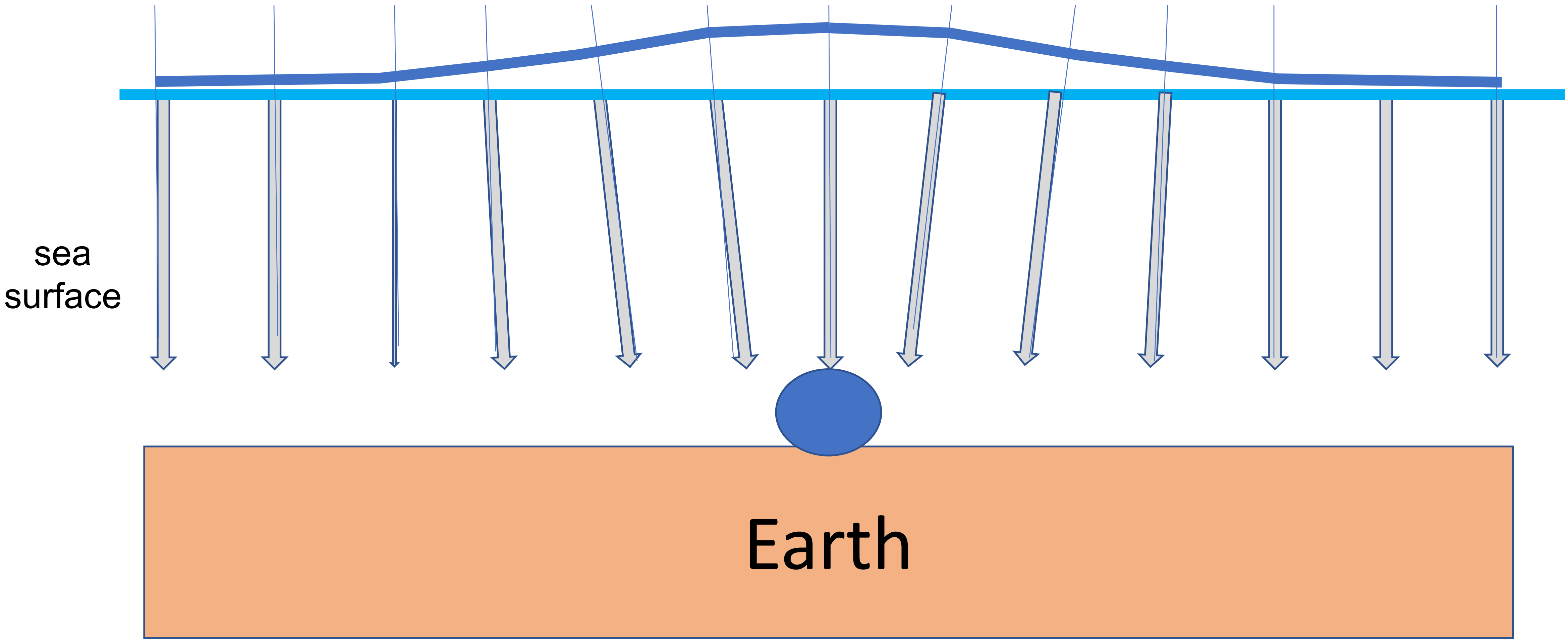
force is the derivative of Enervy




surface perpendicular to field
sea surface
Earth


surface of
equal
energy
sea
surface
Earth

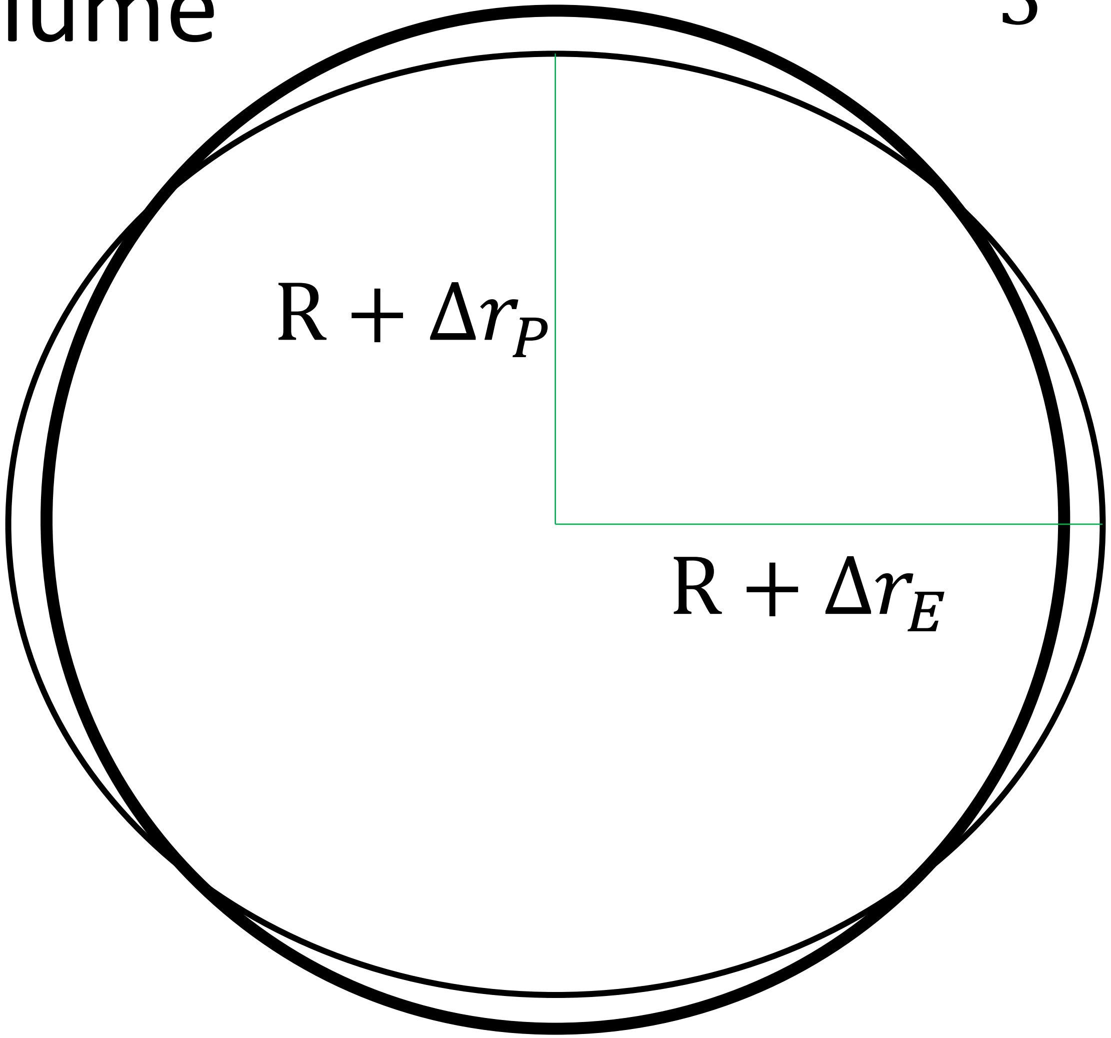
because surface is perpendicular to field,
moving an object along the surface takes no energy



conservation of volume

$$V = \frac{4\pi}{3}(\mathrm{R} + \Delta r_P)(\mathrm{R} + \Delta r_E)(\mathrm{R} + \Delta r_E)$$

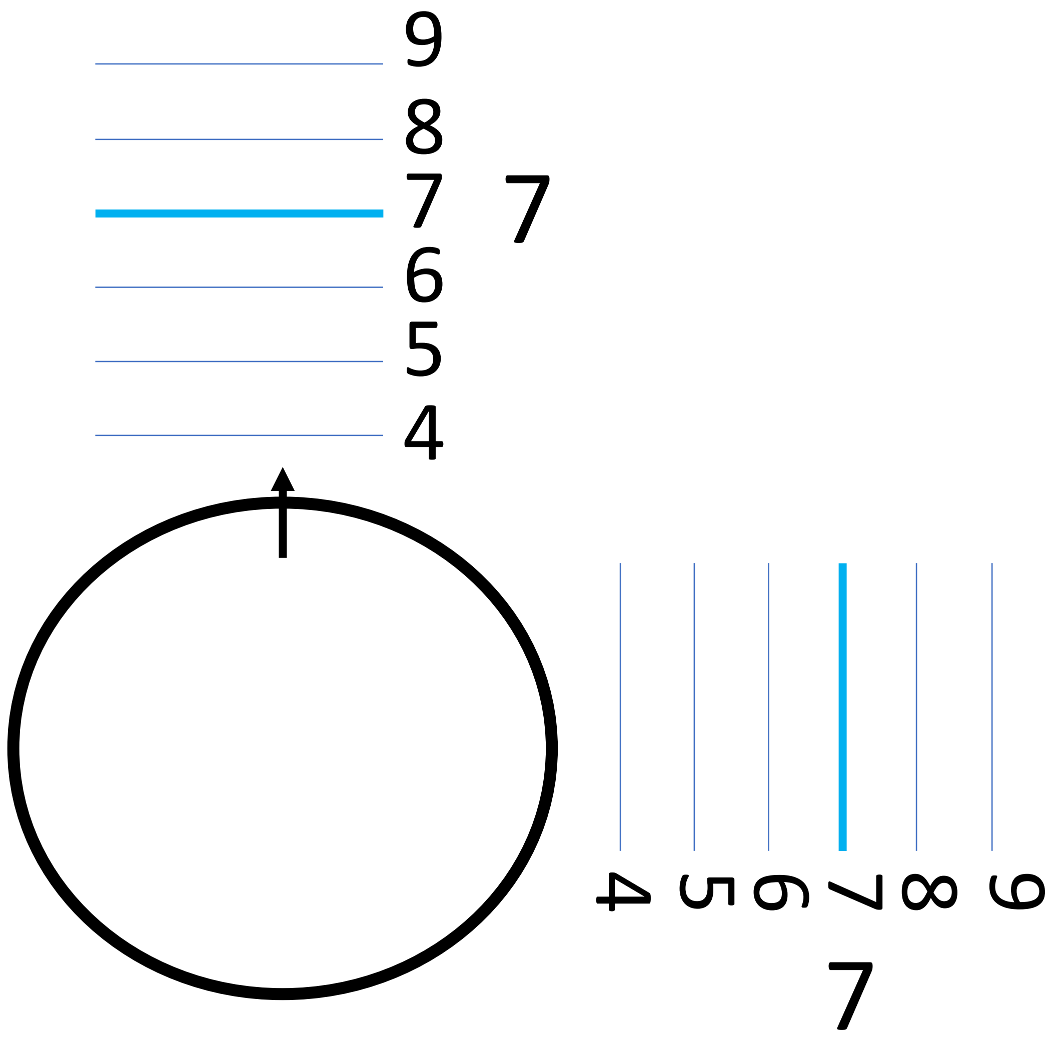
$$\approx \frac{4\pi}{3}R^3\left(1 + \frac{\Delta r_P}{R^2} + \frac{2\Delta r_E}{R^2}\right)$$



$$\frac{\Delta r_P}{R^2} + \frac{2\Delta r_E}{R^2} = 0$$

$$\Delta r_P = -2\Delta r_E$$

assuming Earth is an ellipse

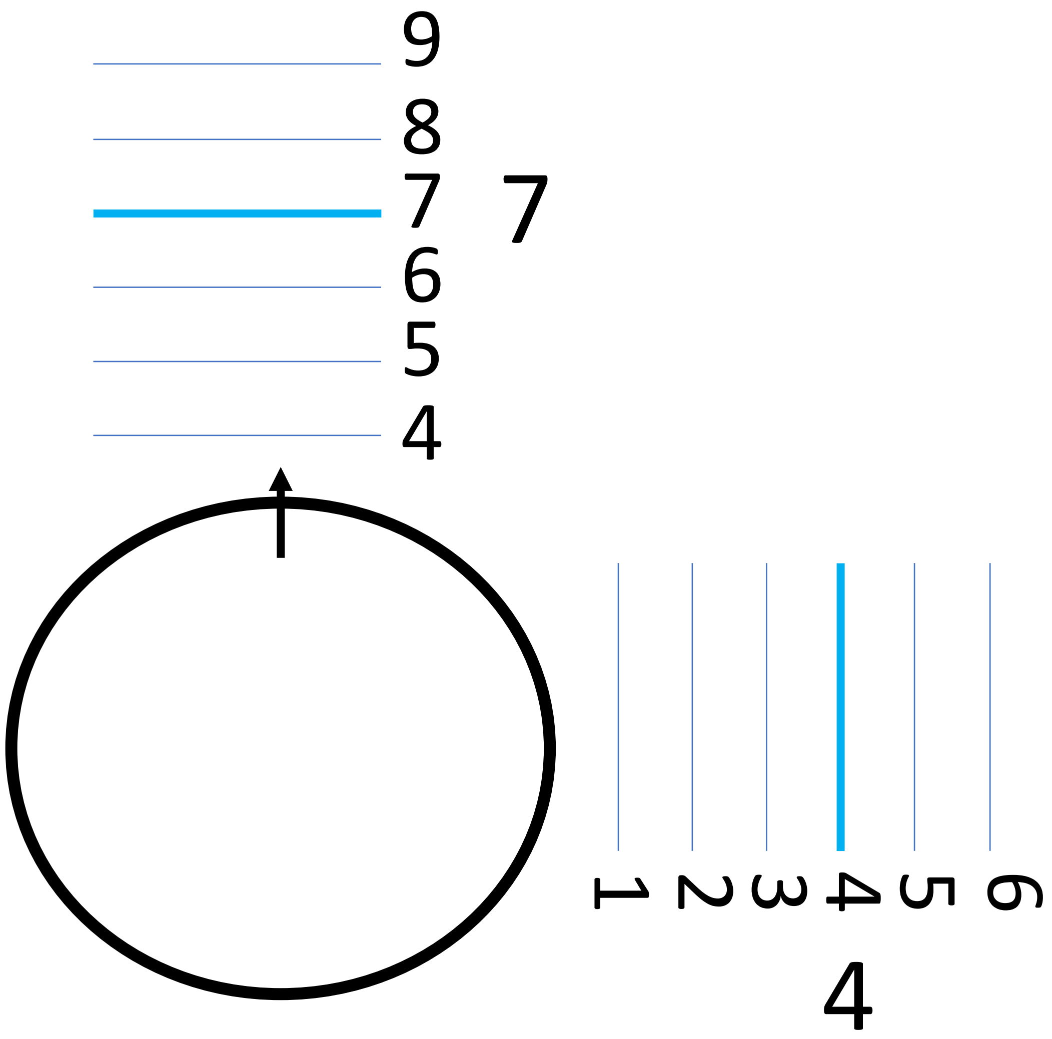


no rotation

energy levels even

water levels even

water energy even

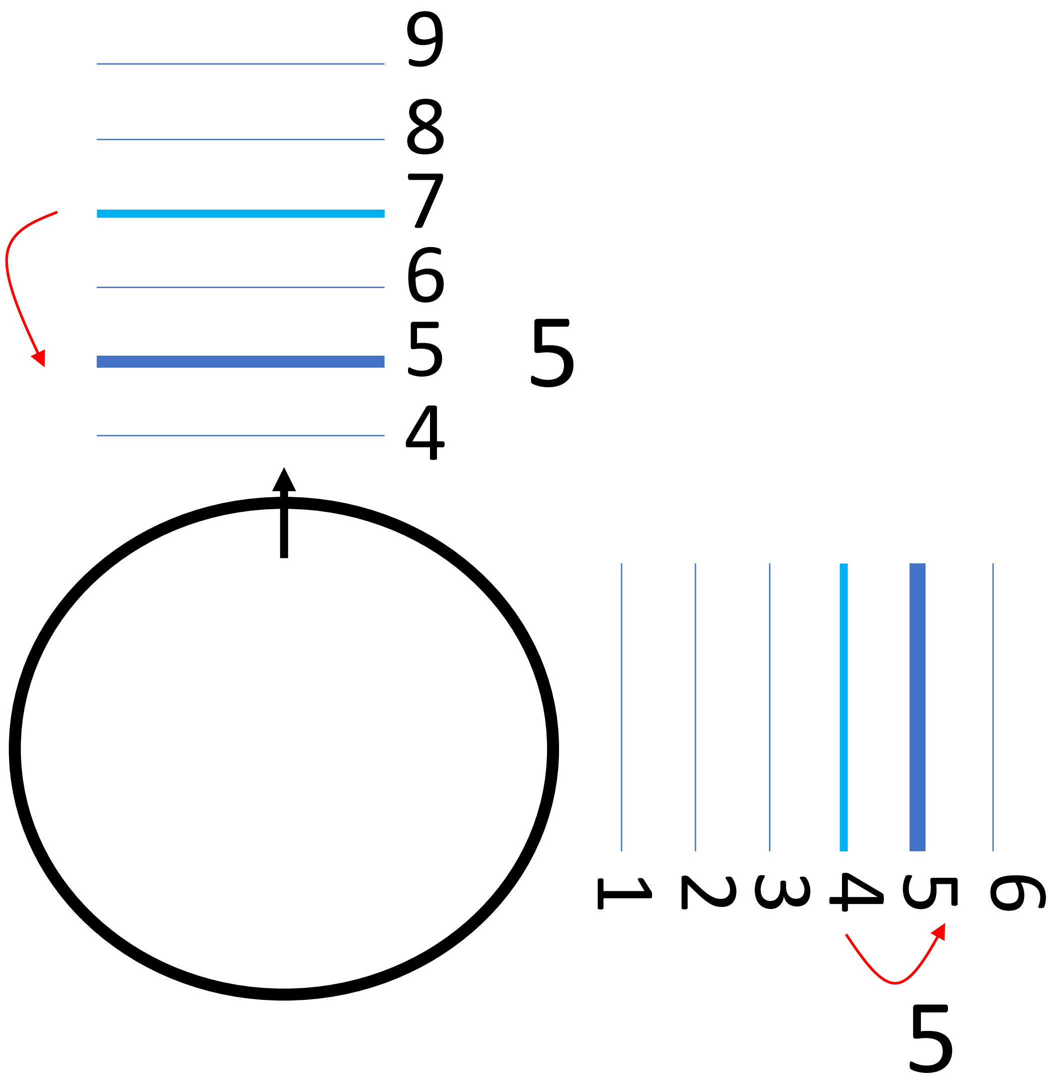


turn on rotation

water levels unchanged

reduces energy levels at equator

energy levels mismatched



rotation is on

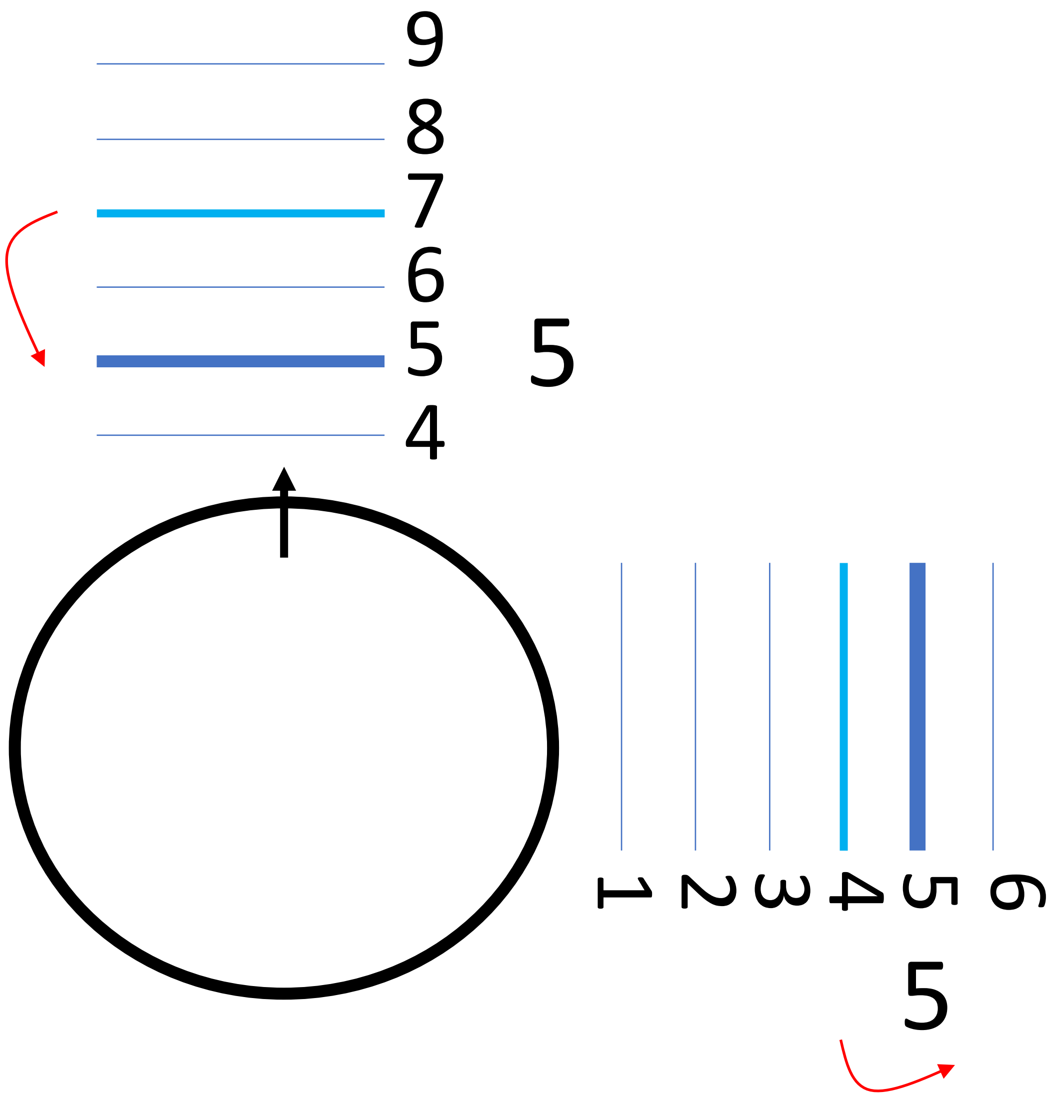
water levels adjust

equator up1

pole down 2

energy levels

now matched



rotation is on

water levels adjust

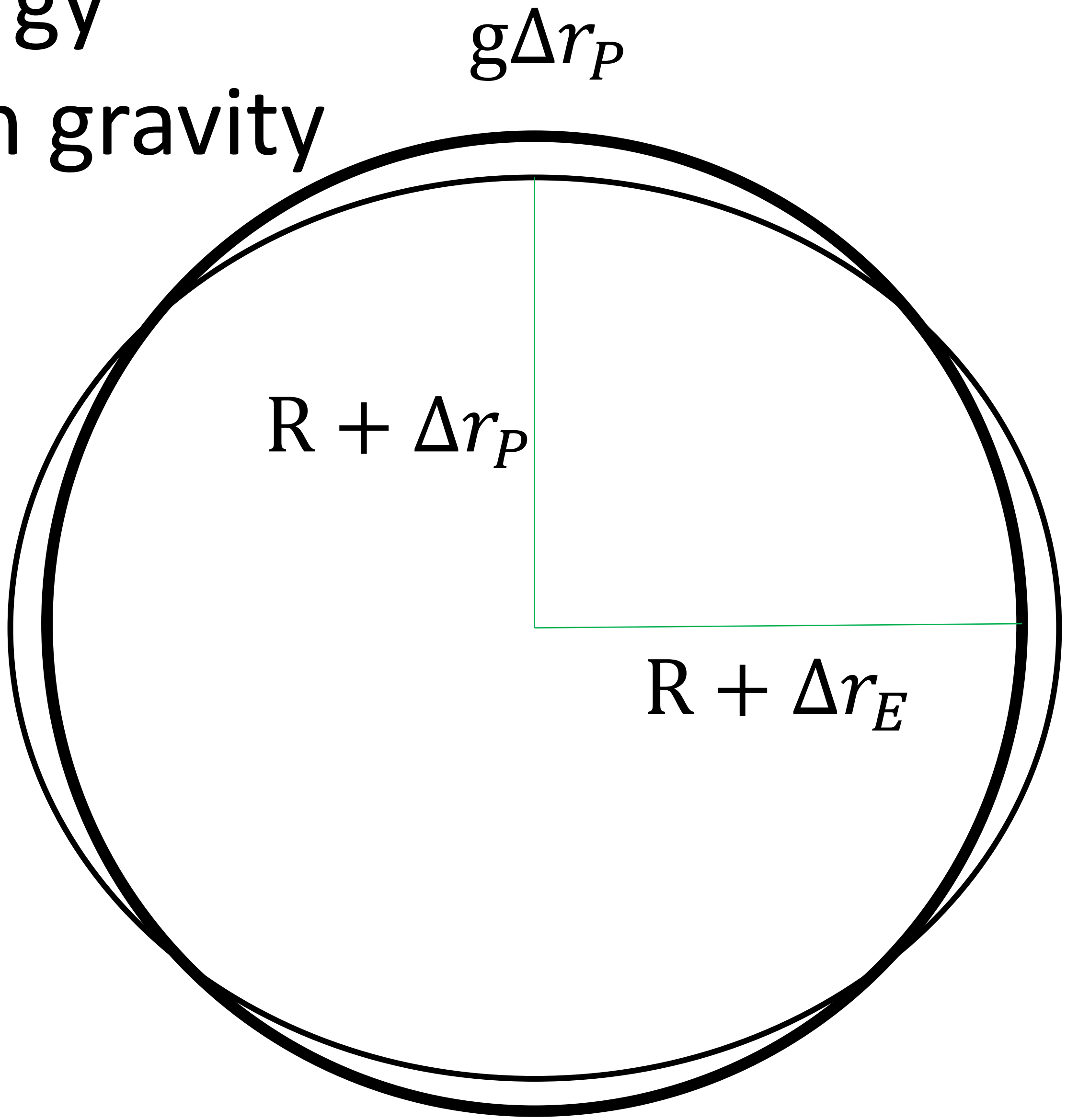
equator up1

pole down 2

energy levels

now matched

energy from gravity



$$f_g = -\gamma M m r^{-2}$$

$$E_g = \gamma M m r^{-1}$$

for equator

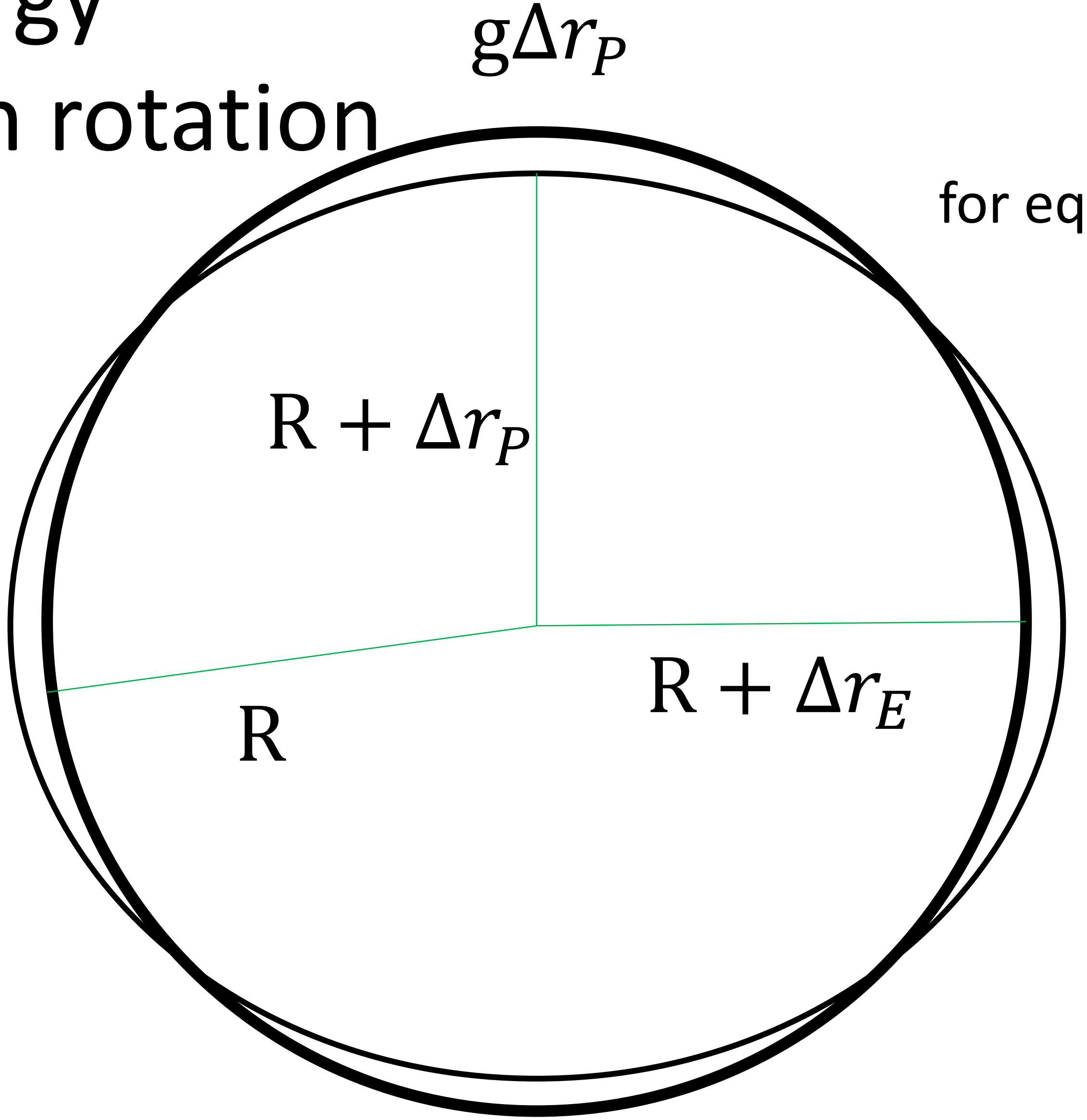
$$E_g = \gamma M m(R + \Delta r_E)^{-1}$$

$$= \frac{\gamma M m}{R}\left(1 + \frac{\Delta r_E}{R}\right)^{-1}$$

$$= \frac{\gamma M m}{R}\left(1 - \frac{\Delta r_E}{R}\right)$$

for pole $E_g = \frac{\gamma M m}{R}\left(1 - \frac{\Delta r_P}{R}\right)$

energy from rotation



$$f_c = m\omega^2 r$$

$$E_c = ½m\omega^2 r^2$$

for equator

$$= ½m\omega^2 (R + \Delta r_E)^2$$

$$= ½m\omega^2 R^2 \left(1 + \frac{\Delta r_E}{R}\right)^2$$

$$= ½m\omega^2 R^2 \left(1 + 2\frac{\Delta r_E}{R}\right)$$

Energy Balance pole equator

$$\frac{\gamma M}{\mathrm{R}}\left(1+\frac{\Delta r_P}{R}\right)=\frac{\gamma M}{\mathrm{R}}\left(1+\frac{\Delta r_E}{R}\right)-½\omega^2R^2\left(1+2\frac{\Delta r_E}{R}\right)$$

$\Delta r_P = -2\Delta r_E$

$$\frac{\gamma M}{\mathrm{R}}\left(1-2\frac{\Delta r_E}{R}\right)-\frac{\gamma M}{\mathrm{R}}\left(1+\frac{\Delta r_E}{R}\right)=-½\omega^2R^2\left(1+2\frac{\Delta r_E}{R}\right)$$

$$\frac{\gamma M}{\mathrm{R}}\left(1+\frac{\Delta r_P}{R}\right)-\frac{\gamma M}{\mathrm{R}}\left(1+\frac{\Delta r_E}{R}\right)=-½\omega^2R^2\left(1+2\frac{\Delta r_E}{R}\right)$$

$$\frac{\gamma Mm}{\mathrm{R}}\left(1-2\frac{\Delta r_E}{R}\right)-\frac{\gamma Mm}{\mathrm{R}}\left(1+\frac{\Delta r_E}{R}\right)=-½m\omega^2R^2\left(1+2\frac{\Delta r_E}{R}\right)$$

cancel   cancel   small

$$\frac{\gamma Mm}{\mathrm{R}}\left(-2\frac{\Delta r_E}{R}\right)-\frac{\gamma Mm}{\mathrm{R}}\left(\frac{\Delta r_E}{R}\right)=-½m\omega^2R^2$$

$$\frac{\gamma M}{\mathrm{R}}\left(1+\frac{\Delta r_P}{R}\right)+\frac{\gamma M}{\mathrm{R}}\left(1+\frac{\Delta r_E}{R}\right)+\tfrac{1}{2}\omega^2R^2\left(1+2\frac{\Delta r_E}{R}\right)=0$$

$$\frac{\gamma Mm}{\mathrm{R}}\left(1-2\frac{\Delta r_E}{R}\right)+\frac{\gamma Mm}{\mathrm{R}}\left(1+\frac{\Delta r_E}{R}\right)=-\tfrac{1}{2}m\omega^2R^2\left(1-2\frac{\Delta r_E}{R}\right)$$

cancel cancel small

$$\frac{\gamma Mm}{\mathrm{R}}\left(-2\frac{\Delta r_E}{R}\right)-\frac{\gamma Mm}{\mathrm{R}}\left(\frac{\Delta r_E}{R}\right)=-\tfrac{1}{2}m\omega^2R^2$$

move to rhs

$$\Delta r_E=\frac{\omega^2R^4}{6\gamma M}$$

$$\Delta r_E = \frac{\omega^2 R^4}{6\gamma M} = \frac{\left(\frac{2\pi}{T}\right)^2 R^4}{6\gamma M} \approx 4 \text{ km}$$

$$M = 6 \times 10^{24} \text{ kg}$$

$$T = 86400\, s$$

$$R = 6.37 \times 10^6\, m$$

$$\gamma = 6.67 \times 10^{-11}\, m^3 kg^{-1}\, s^{-2}$$

$$units:\ s^{-2} m^4 \frac{kg s^2}{m^3} \frac{1}{kg} = m$$

	A	B
1	M	6.00E+24
2	T	86400
3	R	6.37E+06
4	gamma	6.67E-11
5	w	7.2722E-05
6		
7	DrE m	3.63E+03
8	dE km	3.63
9		
10		

$\Delta r_E \approx 4 \text{ km}$

$\Delta r_P \approx -8 \text{ km}$

predicted

total bulge $\approx 12 \text{ km}$

total bulge (observed) $\approx$

8 km + 14 km = 24 km

(Pretty accurate for a simple calculation)