Solid Earth Dynamics

Bill Menke, Instructor

Lecture 26

Solid Earth Dynamics

Today is the Last Formal Lecture

Review Session Thursday
Come with questions …

Solid Earth Dynamics

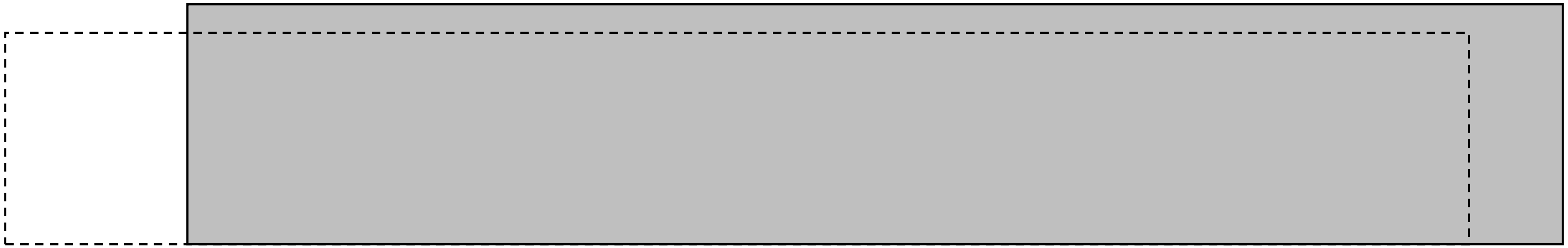
Glaciology
Crevasses


Crevasses

need extensional horizontal stress to open crevasse

case 1: Compressing glacier, $r < 0$

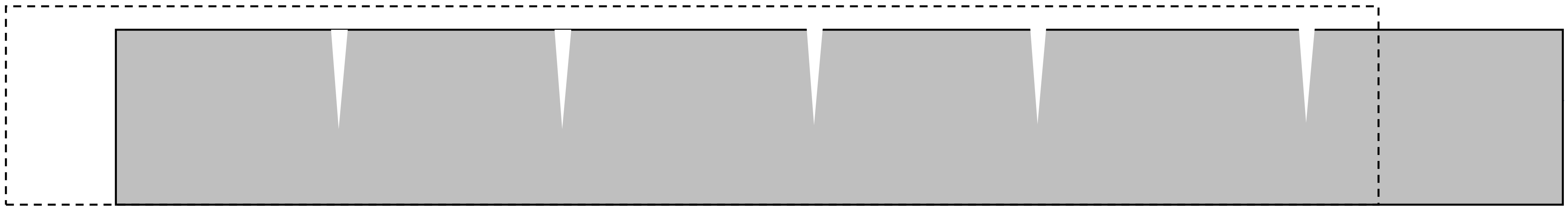
case 1: Glacier thickens



case 1: Extending glacier, $r > 0$

case 1: Extending glacier, $r < 0$

Crevasses form



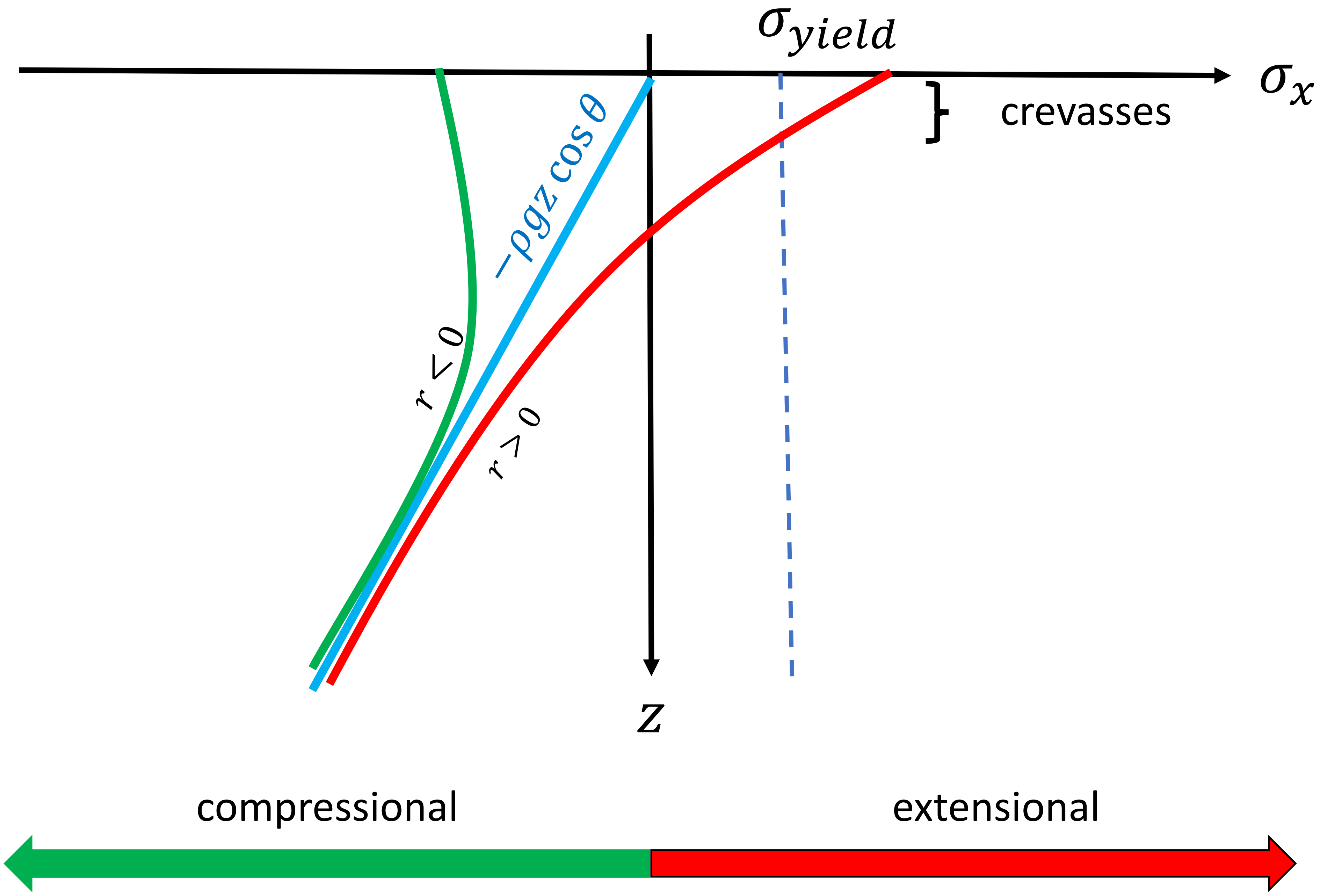
Questions that we will answer today

What must the extension rate be to get crevasses?

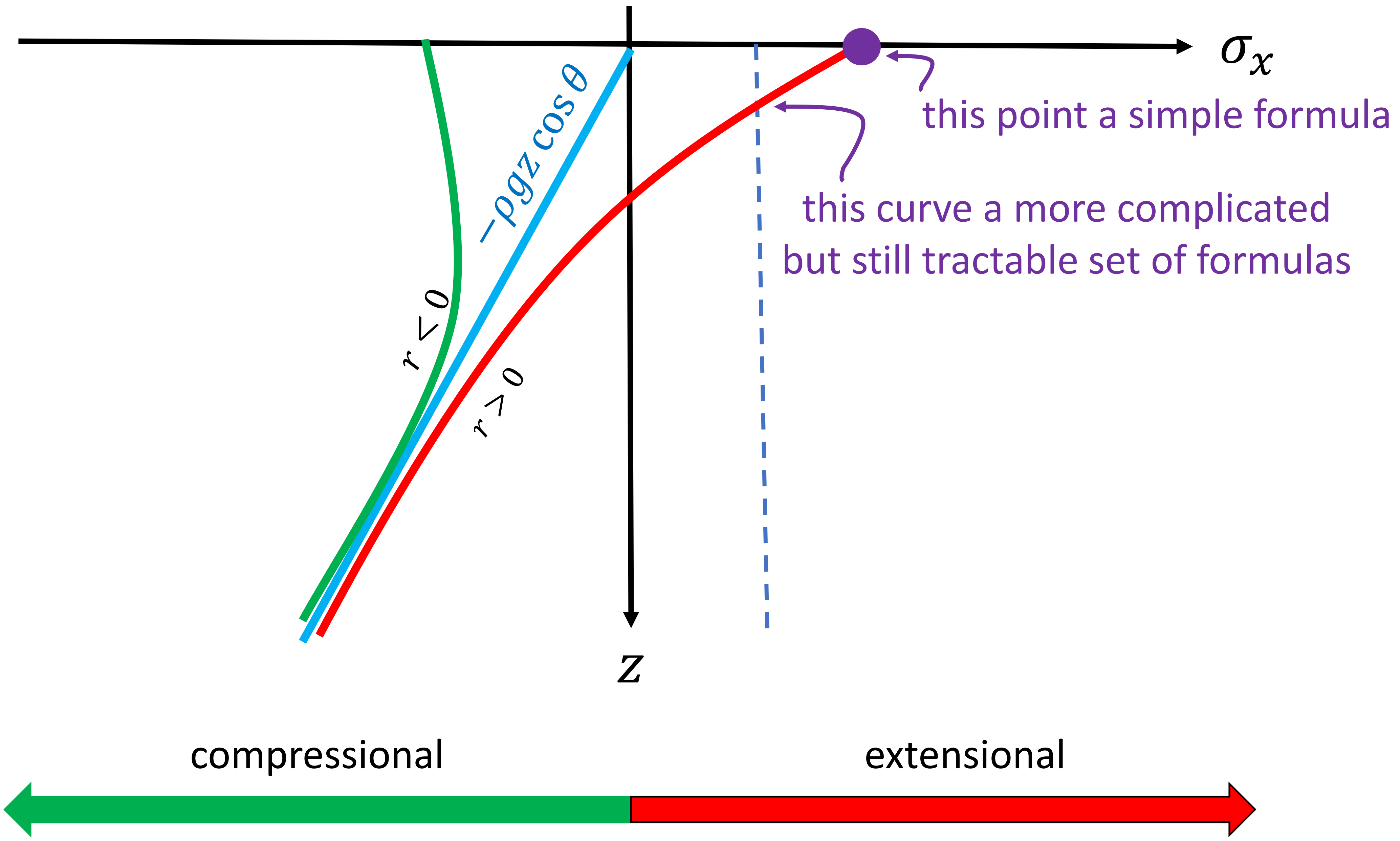
How deep into the glacier will the crevasses extend?

First the answer …

… then the analysis



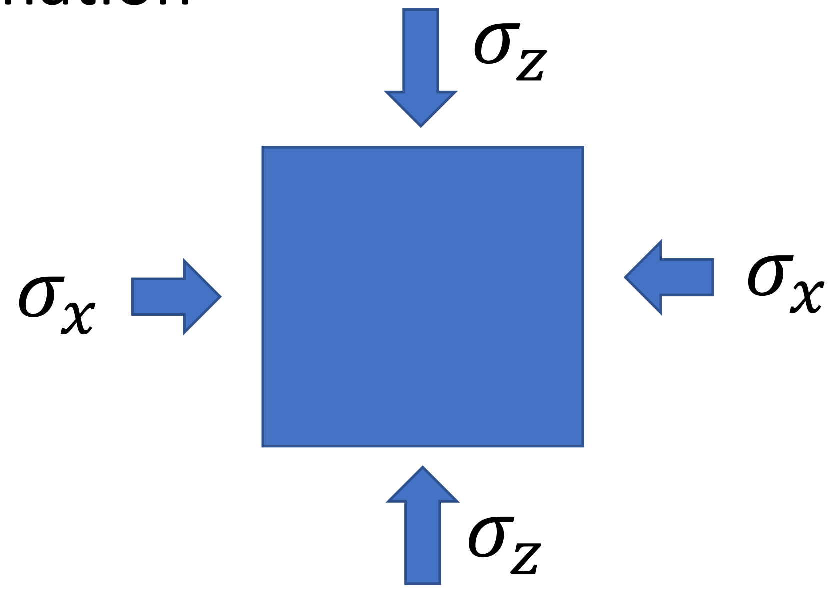
σ_{yield}
σ_x
crevasses
$-\rho g z \cos\theta$
$r < 0$
$r > 0$
z
compressional
extensional



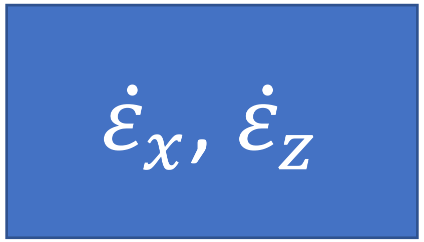
σ_x
this point a simple formula
this curve a more complicated
but still tractable set of formulas
$-\rho g z \cos\theta$
$r < 0$
$r > 0$
z
compressional
extensional

Modes of deformation

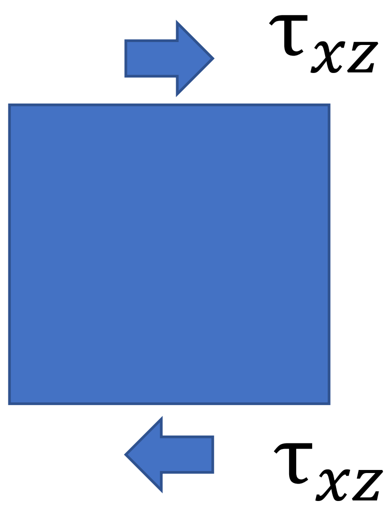
differences in normal stresses



deforms to



shear stress



deforms to

Lab observation:

flow law independent of overall pressure, $-p = \frac{\sigma_x + \sigma_z}{2}$

Solution, use effective stress

$$\sigma'_x = \sigma_x + p = \sigma_x - \frac{\sigma_x + \sigma_z}{2}$$
$$\sigma'_z = \sigma_x + p = \sigma_z - \frac{\sigma_x + \sigma_z}{2}$$

Lab observation:

flow law independent of overall pressure, $-p = \frac{\sigma_x + \sigma_z}{2}$

Solution, use effective stress

$$\sigma'_x = \sigma_x - \frac{\sigma_x + \sigma_z}{2} = \frac{\sigma_x - \sigma_z}{2}$$

$$\sigma'_z = \sigma_z - \frac{\sigma_x + \sigma_z}{2} = -\sigma'_x$$

flow law when both normal and shear stresses occur

flow law when both normal and shear stresses occur

$$\dot{\varepsilon}_x = (\text{effective viscosity})\sigma'_{\mathrm{x}}$$

$$\dot{\varepsilon}_{xz} = (\text{effective viscosity})\tau_{xz}$$

flow law when both normal and shear stresses occur

flow law when both normal and shear stresses occur

$$\dot{\varepsilon}_x = (\text{effective viscosity})\sigma'_{\mathrm{x}}$$

$$\dot{\varepsilon}_{xz} = (\text{effective viscosity})\tau_{xz}$$

depends on
the overall
state of stress

flow law when both normal and shear stresses occur

third power flow law when both normal and shear stresses occur

$$\dot{\varepsilon}_x = A\tau^2\sigma'_x \qquad \dot{\varepsilon}_{xz} = A\tau^2\tau_{xz}$$

with τ a measure of the overall state of stress

$$\tau = \sqrt{½(\sigma'_x)^2 + ½(\sigma'_z)^2 + (\tau_{xz})^2}$$

$$\tau^2 = ½(\sigma'_x)^2 + ½(\sigma'_z)^2 + (\tau_{xz})^2$$

sometimes called the "stress invariant"

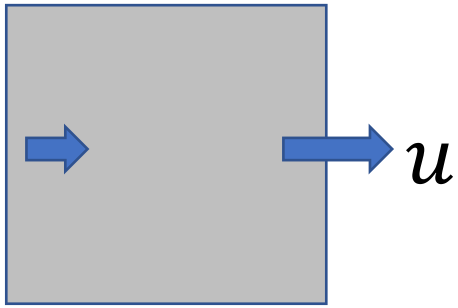
also define $\dot{\varepsilon}$ as a measure of the overall state of strain rate

$$(\dot{\varepsilon})^2 = ½(\dot{\varepsilon}_x)^2 + ½(\dot{\varepsilon}_z)^2 + (\dot{\varepsilon}_{xz})^2$$

sometimes called the “starin-rate invariant”

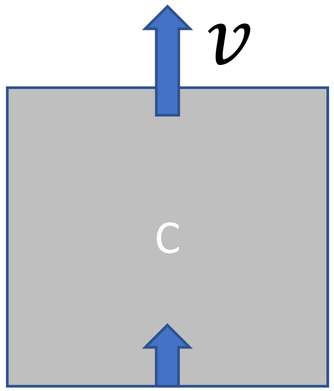
it can be shown that $\dot{\varepsilon}$ and τ obey the flow law

$$\dot{\varepsilon} = A\tau^3$$



deforms to

$$\dot{\varepsilon}_x = \frac{du}{dx}$$



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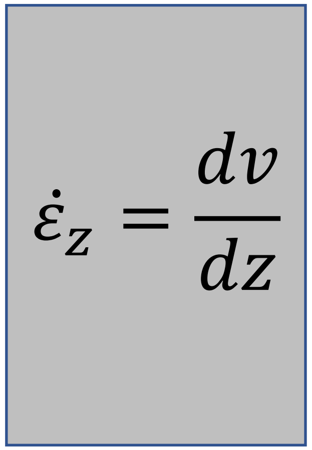

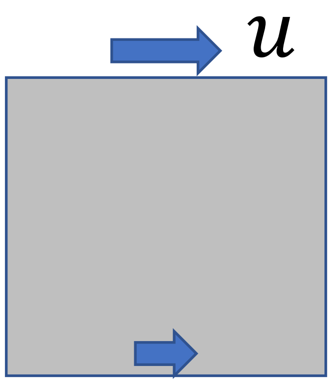

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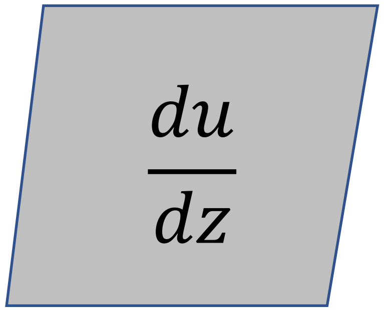
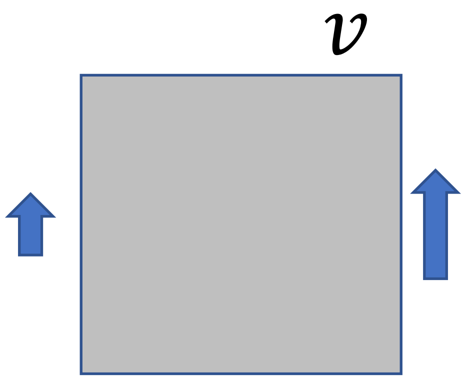
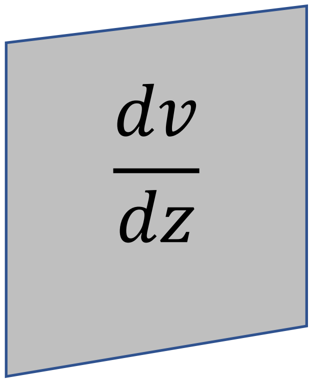


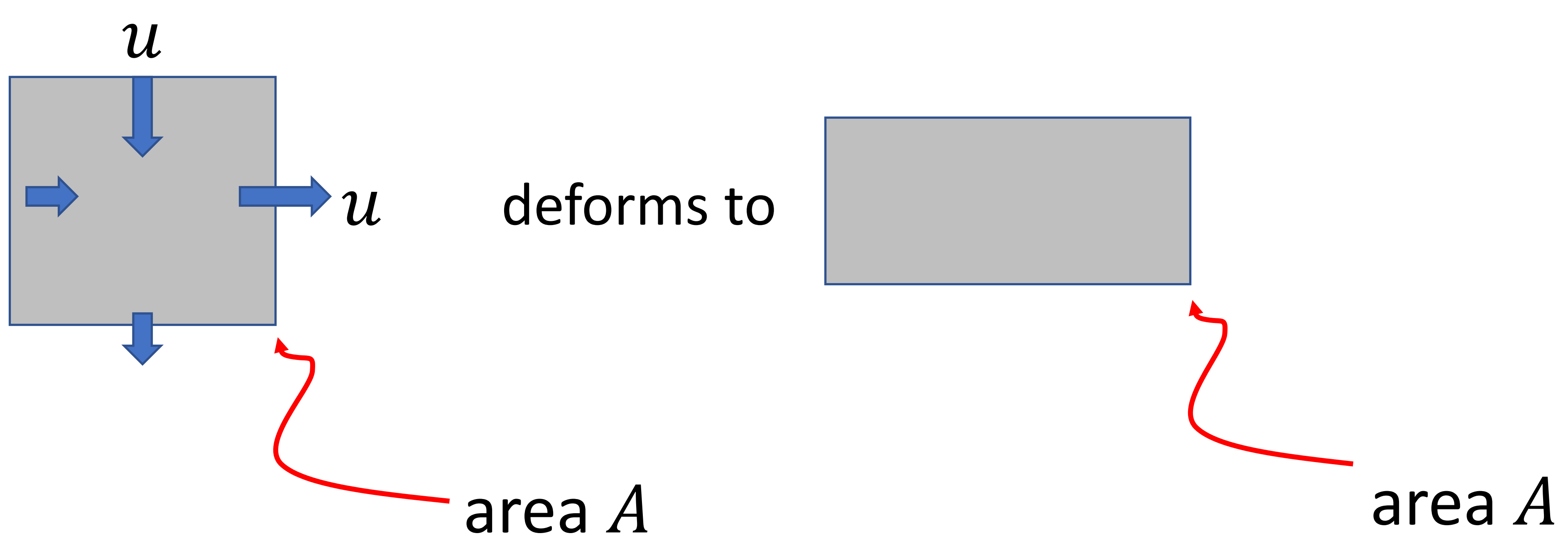


deforms to



$$\dot{\varepsilon}_{xz} = ½\frac{du}{dz} + ½\frac{dv}{dx}$$

factor of ½ needed to remove effect of rotation



conservation of volume: areas are equal

$$\frac{du}{dx} = -\frac{dv}{dz}$$

GOAL

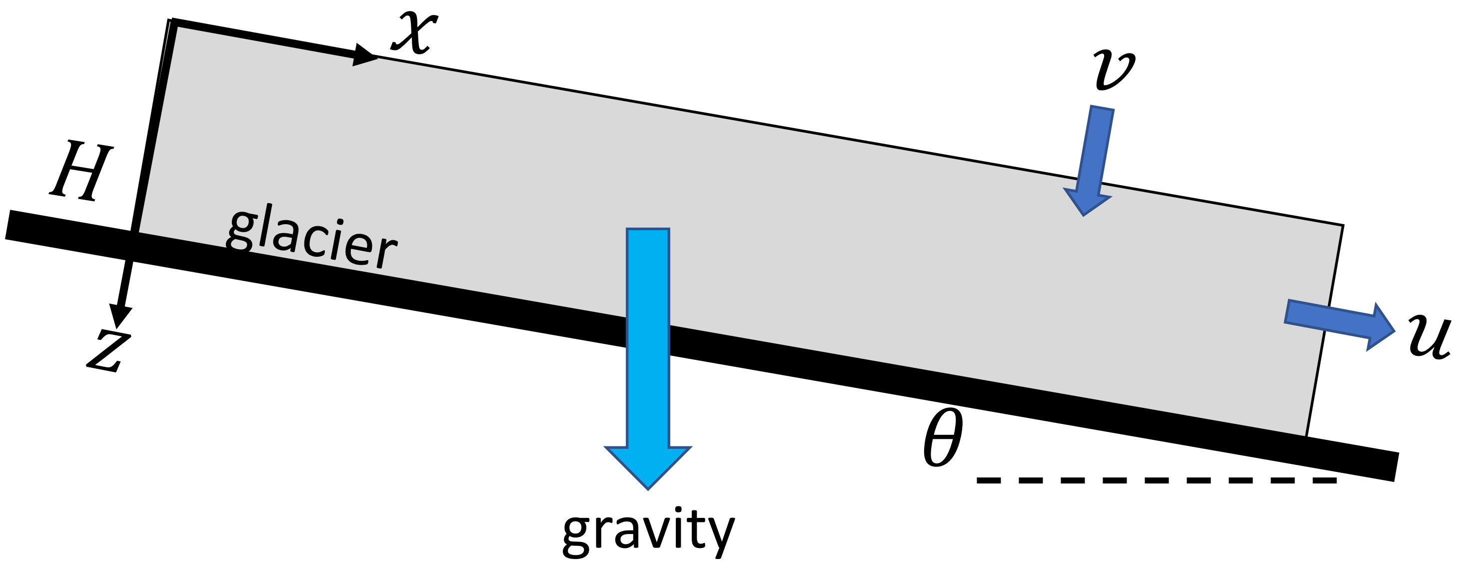
Understand stress in a glacier that is expanding or contracting

relate that to crevassing

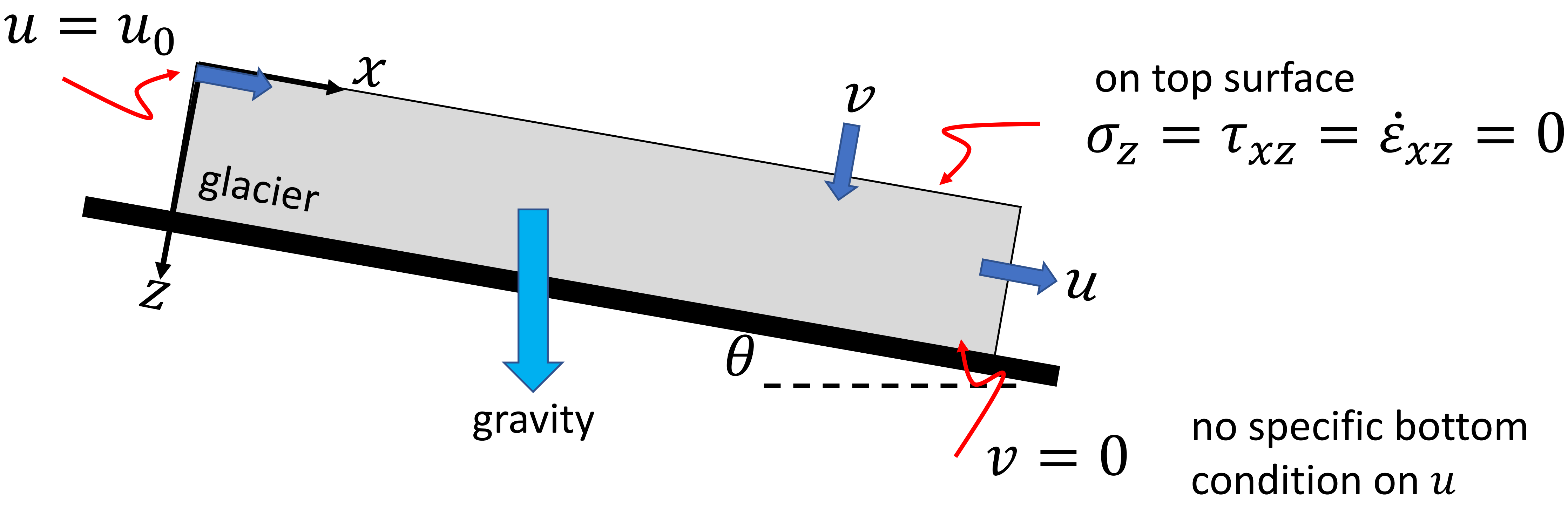
so primarily interested in σ_x

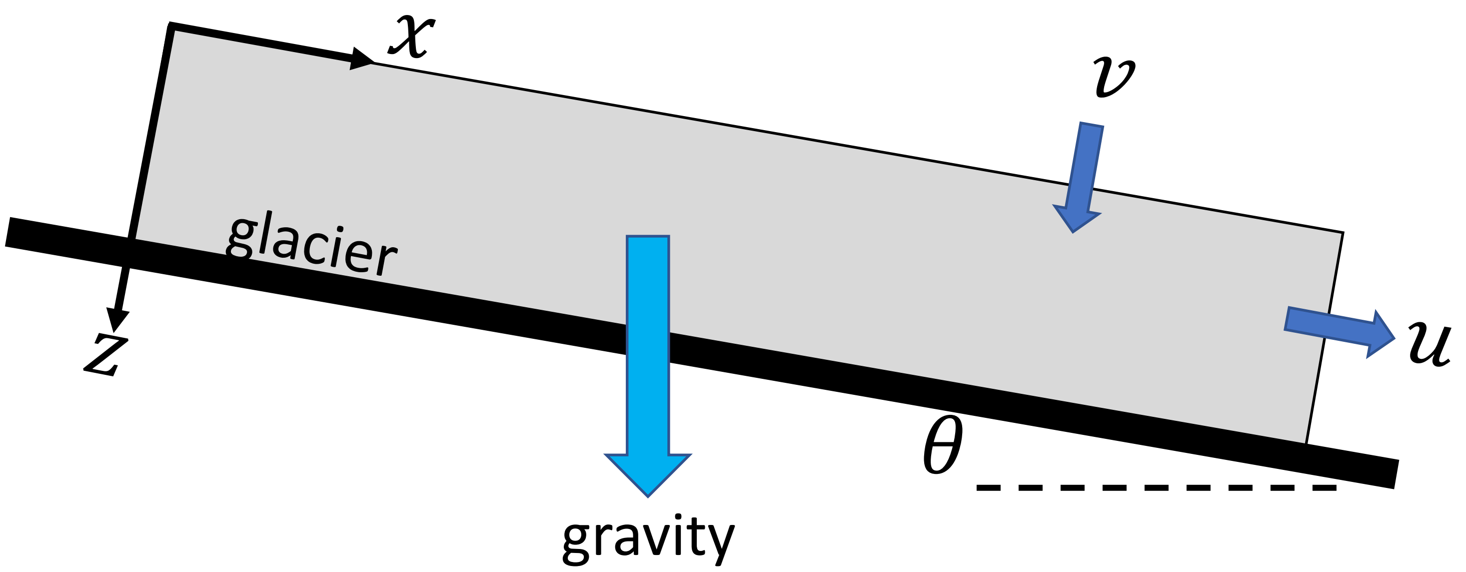
MODEL SETEP

model of glacier with variable thickness



model of glacier with variable thickness



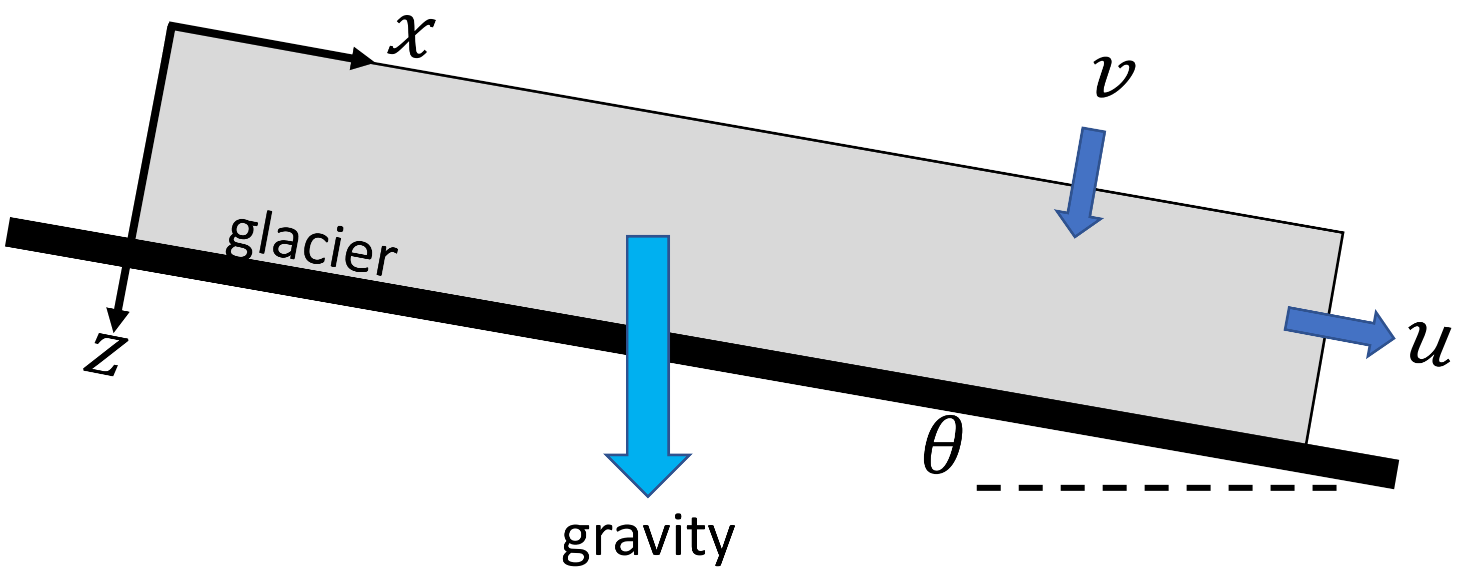


newton's law

$$\begin{cases} \dfrac{d\sigma_x}{dx} + \dfrac{d\tau_{xz}}{dz} = -\sigma g \sin\theta \\ \dfrac{d\tau_{xz}}{dx} + \dfrac{d\sigma_z}{dz} = -\sigma g \cos\theta \end{cases}$$

state of stress

$$4\tau^2 = (\sigma_x - \sigma_z)^2 + 4\tau_{xy}^2$$



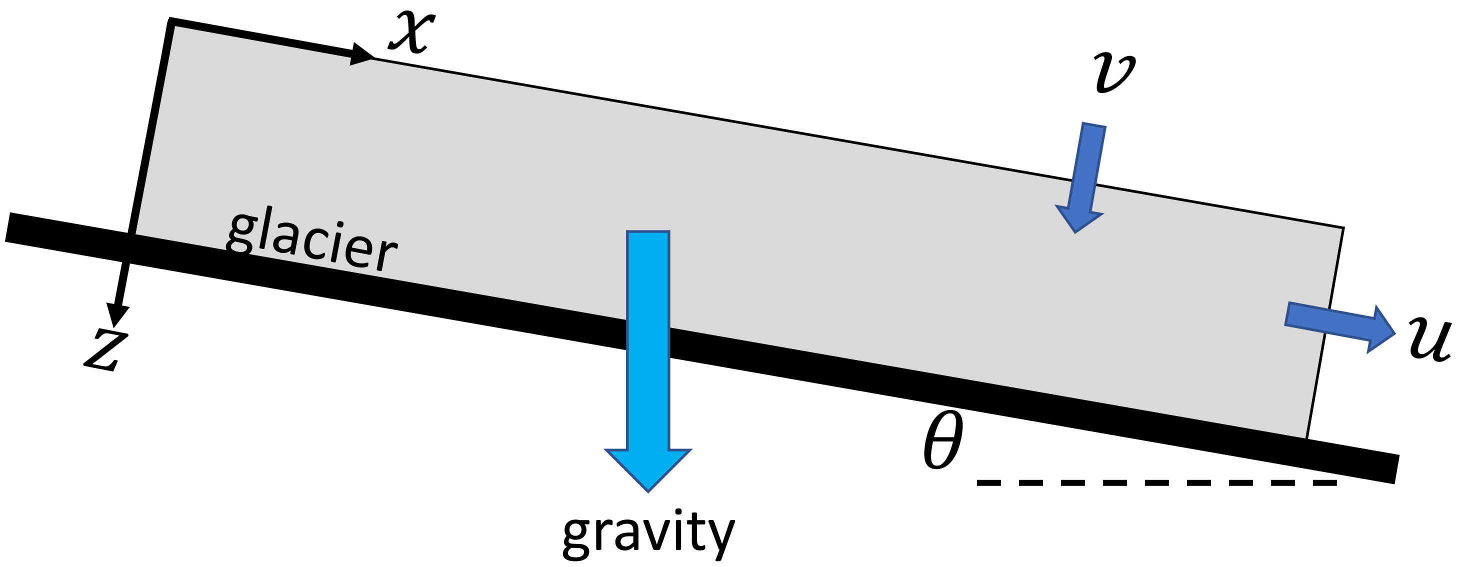
flow law

$$\dot{\varepsilon}_x = \frac{du}{dx} = A\tau^2 \tfrac{1}{2}(\sigma_x - \sigma_z)$$

$$\dot{\varepsilon}_{xy} = \tfrac{1}{2}\left(\frac{du}{dz} + \frac{dv}{dx}\right) = A\tau^2 \tau_{xy}$$

state of strain-rate

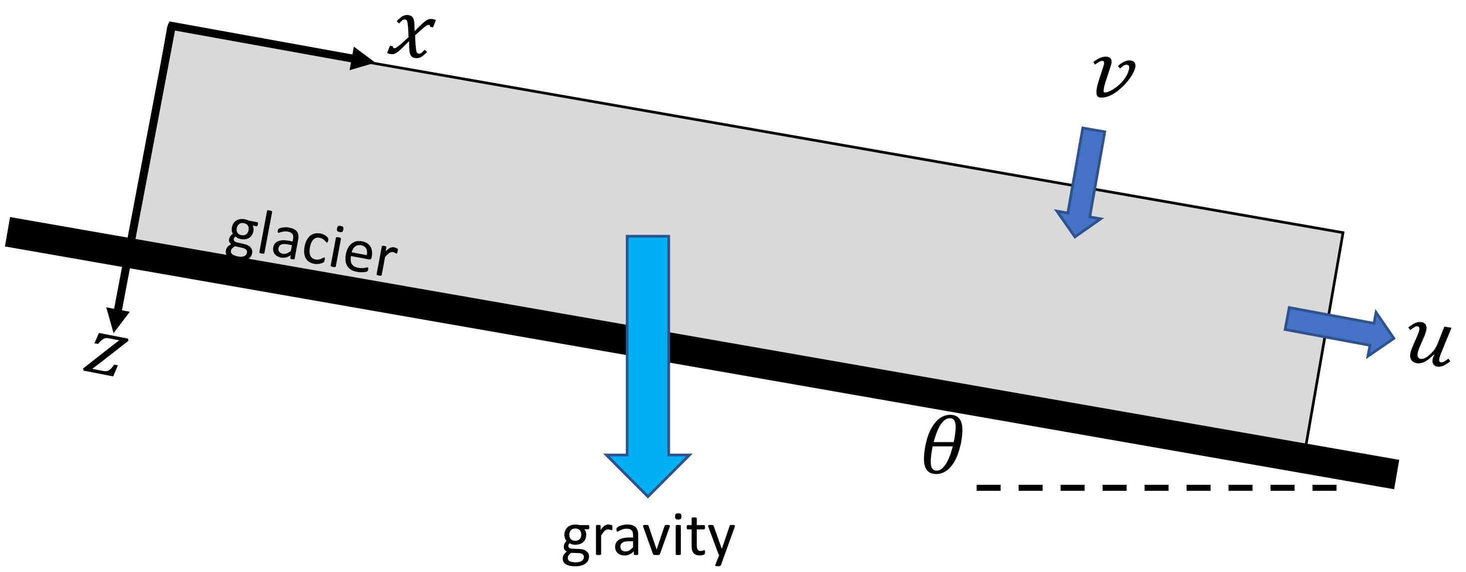
$$2\dot{\varepsilon}^2 = \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dz}\right)^2 + \tfrac{1}{2}\left(\frac{du}{dz} + \frac{dv}{dx}\right)^2$$



Approximation:

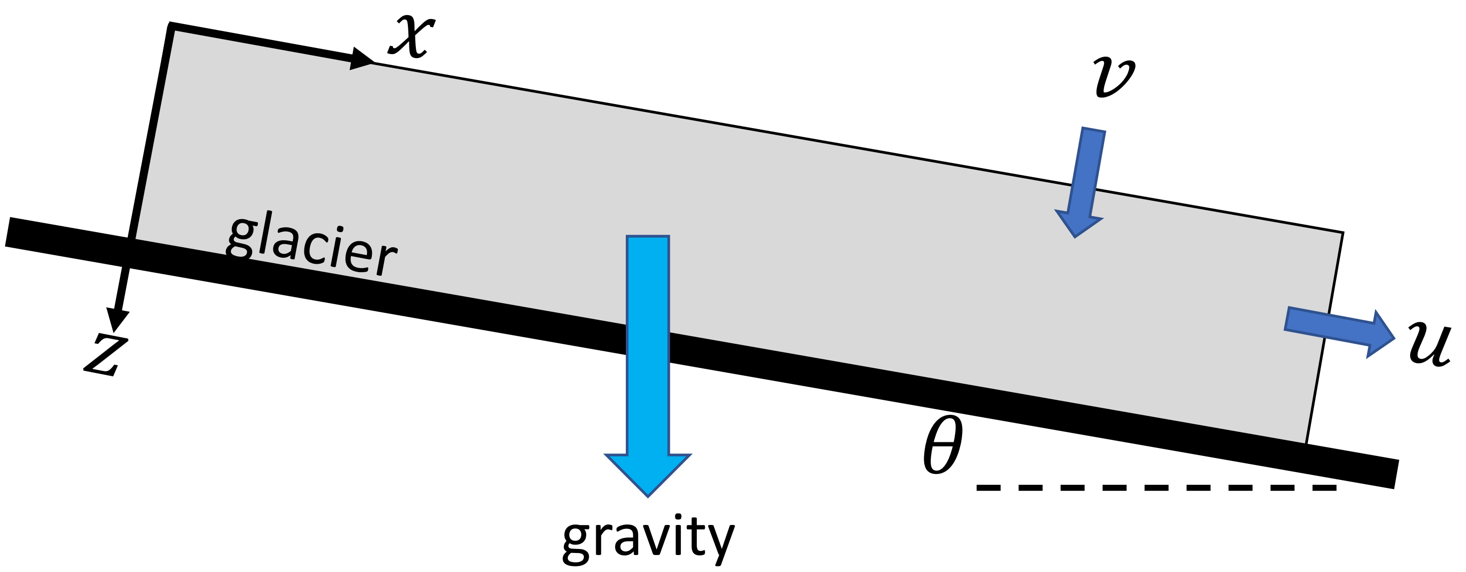
Long glacier, so all stresses independent of x

Variation of Stress with Position



newton's law

$$\begin{cases} \dfrac{d\sigma_x}{dx} + \dfrac{d\tau_{xz}}{dz} = -\rho g \sin\theta \\[2ex] \dfrac{d\tau_{xz}}{dx} + \dfrac{d\sigma_z}{dz} = -\rho g \cos\theta \end{cases}$$

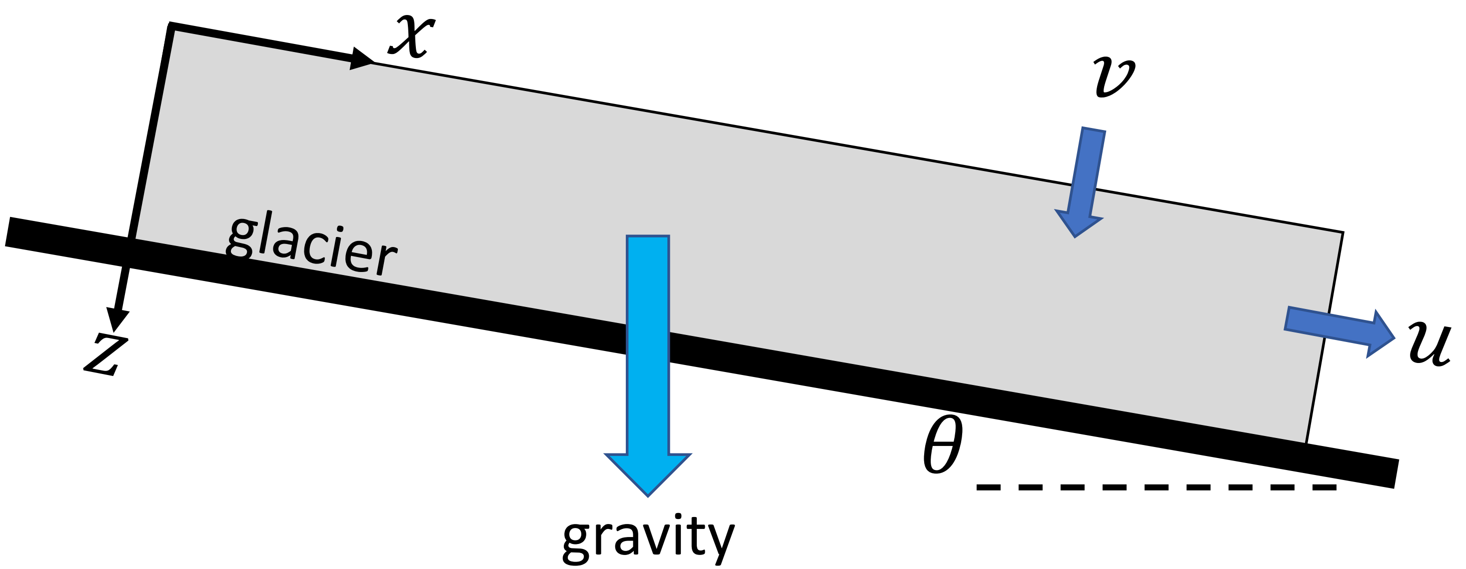


Goal: deduce σ_x

newton's law

$$\frac{d\sigma_x}{dx} + \frac{d\tau_{xz}}{dz} = -\rho g \sin\theta$$ implies $\tau_{xz} = -\rho g z \sin\theta$

$$\frac{d\tau_{xz}}{dx} + \frac{d\sigma_z}{dz} = -\rho g \cos\theta$$ implies $\sigma_z = -\rho g z \cos\theta$



Goal: deduce σ_x

solution of Newton's law

$$\tau_{xz} = -\rho g z \sin\theta$$

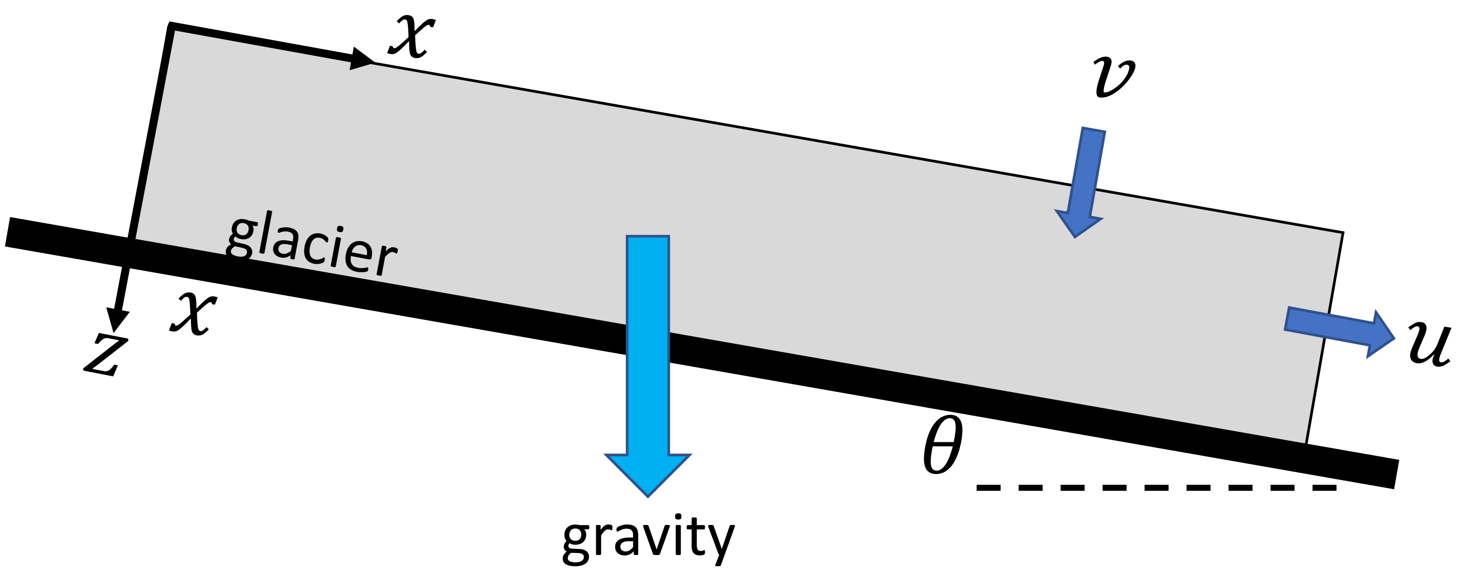
$$\sigma_z = -\rho g z \cos\theta$$

state of stress $4\tau^2 = (\sigma_x - \sigma_z)^2 + 4\tau_{xy}^2$

quadratic equation for σ_x

$$4\tau^2 = \sigma_x^2 - 2\sigma_z\sigma_x + \sigma_z^2 + 4\tau_{xy}^2$$

$$0 = \underbrace{1}_{A}\sigma_x^2 - \underbrace{2\sigma_z}_{B}\sigma_x + \underbrace{\sigma_z^2 + 4\tau_{xy}^2 - 4\tau^2}_{C}$$



solution of Newton's law

$$\tau_{xz} = -\rho g z \sin\theta$$

$$\sigma_z = -\rho g z \cos\theta$$

state of stress $$4\tau^2 = (\sigma_x - \sigma_z)^2 + 4\tau_{xy}^2$$

quadratic equation for σ_x

$$4\tau^2 = \sigma_x^2 - 2\sigma_z\sigma_x + \sigma_z^2 + 4\tau_{xy}^2$$

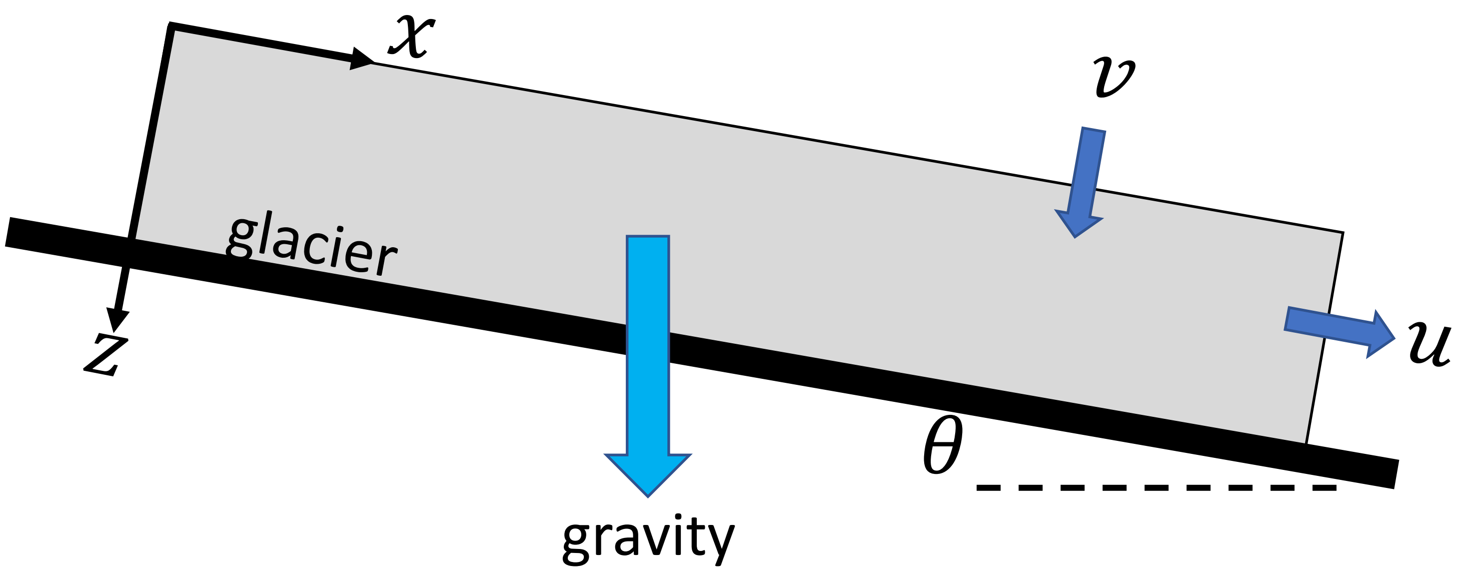
$$0 = 1\sigma_x^2 - 2\sigma_z\sigma_x + \sigma_z^2 + 4\tau_{xy}^2 - 4\tau^2$$

A B C

$$B = -2\sigma_z$$

$$B^2 - 4AC =$$

$$= 4\sigma_z^2 - 4\sigma_z^2 - 16\tau_{xy}^2 + 16\tau^2$$



quadratic formula for σ_x

$$-\tfrac{1}{2}B \pm \tfrac{1}{2}\sqrt{B^2 - 4AC}$$

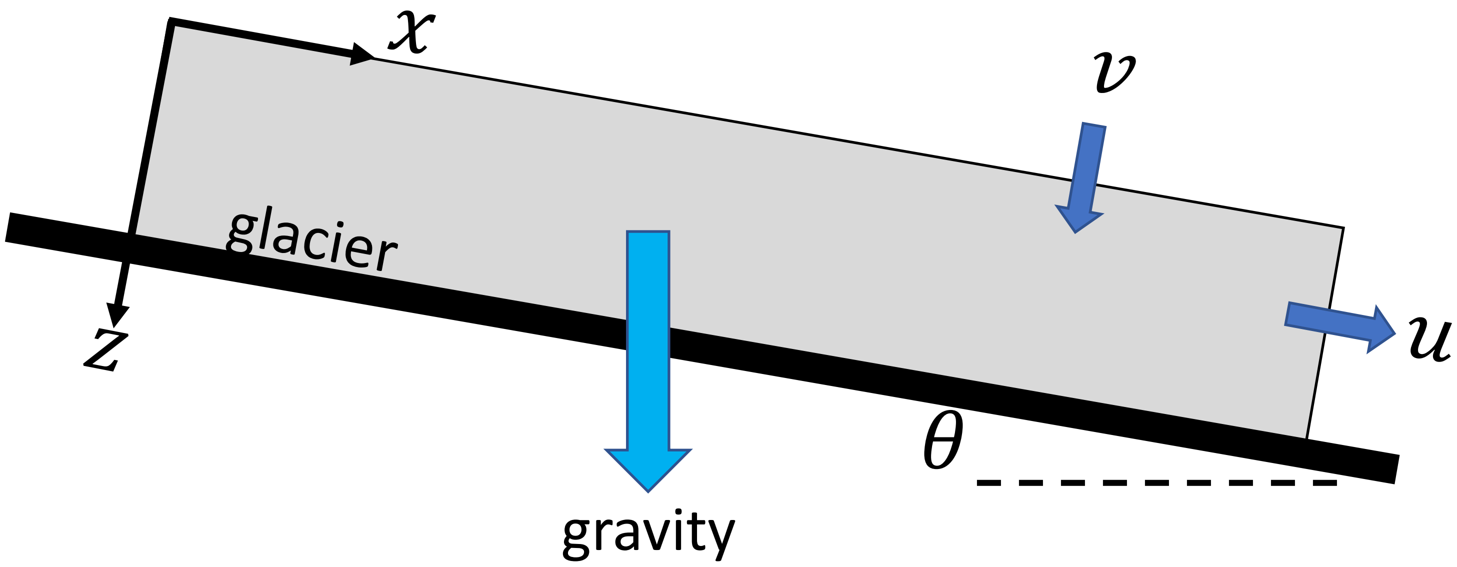
$$B = -2\sigma_z$$

$$\sigma_x = \sigma_z \pm 2\sqrt{\tau^2 - \tau_{xy}^2}$$

$$B^2 - 4AC =$$

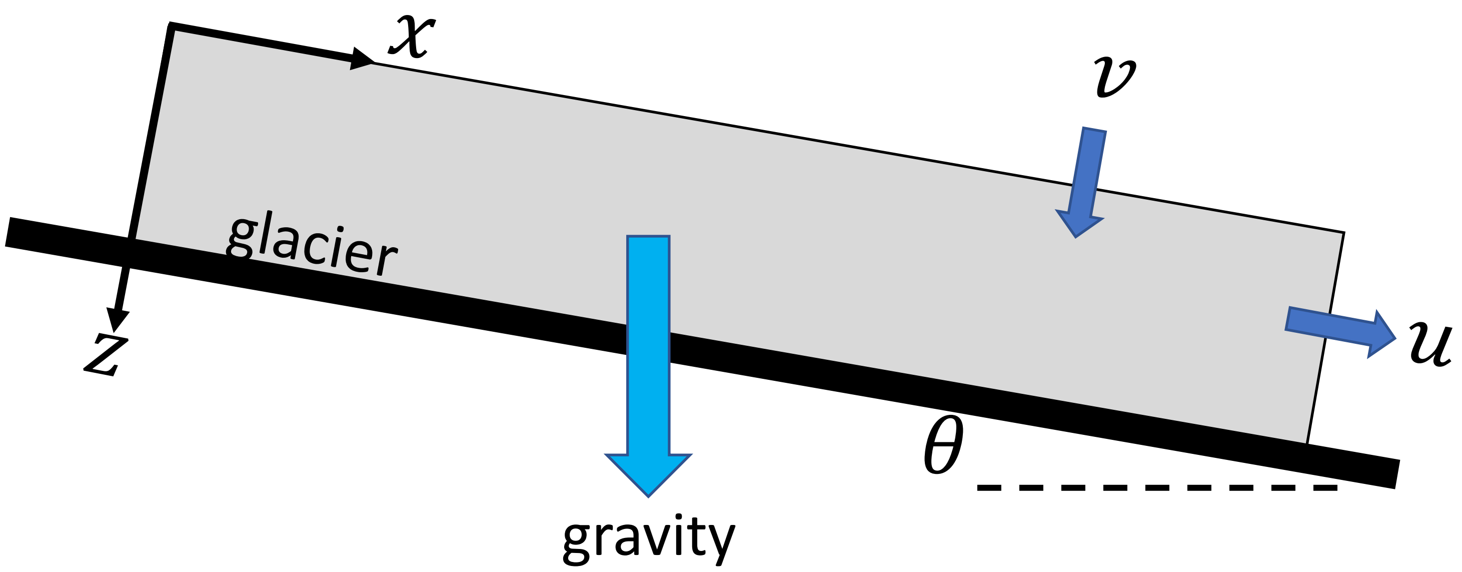
$$= 16\tau^2 - 16\tau_{xy}^2$$

$$\sigma_x = -\rho g z \cos\theta \pm 2\sqrt{\tau^2 - (\rho g z \sin\theta)^2}$$



$$\sigma_x = -\rho g z \cos\theta \pm 2\sqrt{\tau^2 - (\rho g z \sin\theta)^2}$$

still need tp
figure out τ

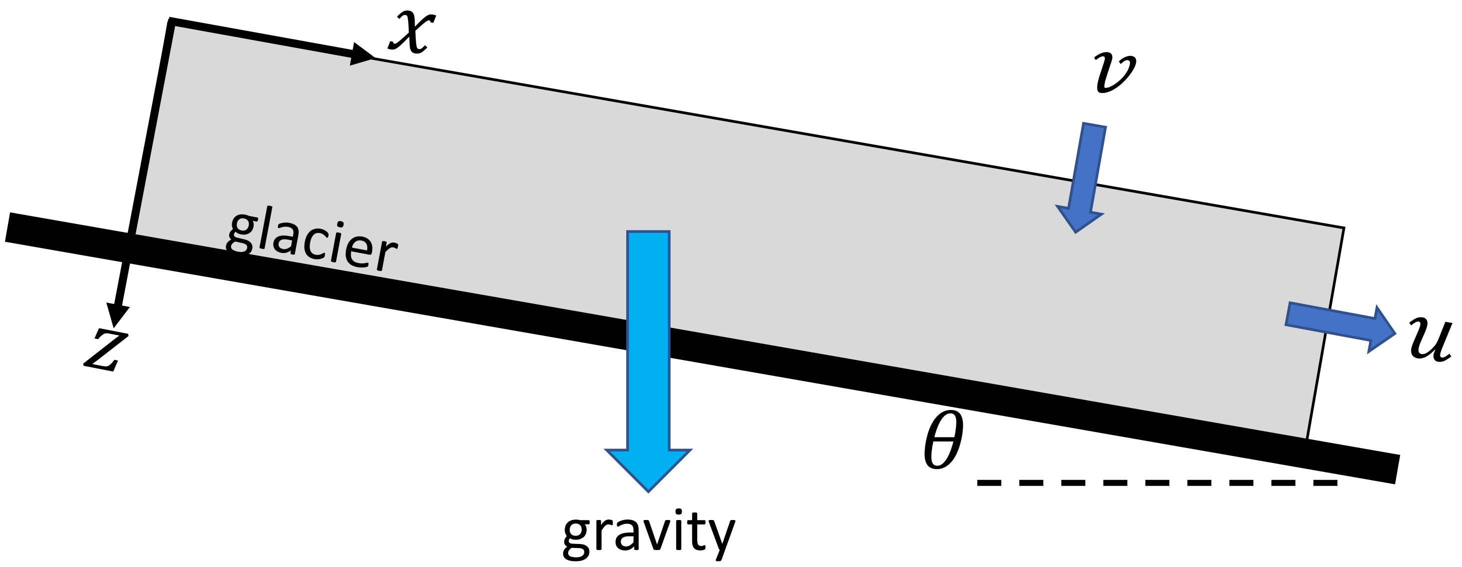


$$\sigma_x = -\rho g z \cos\theta \pm 2\sqrt{\tau^2 - (\rho g z \sin\theta)^2}$$

two solutions

$$\sigma_x = -\rho g z \cos\theta - 2\sqrt{\tau^2 - (\rho g z \sin\theta)^2}$$ Compressive

$$\sigma_x = -\rho g z \cos\theta + 2\sqrt{\tau^2 - (\rho g z \sin\theta)^2}$$ Possibly Extensional (crevasses)

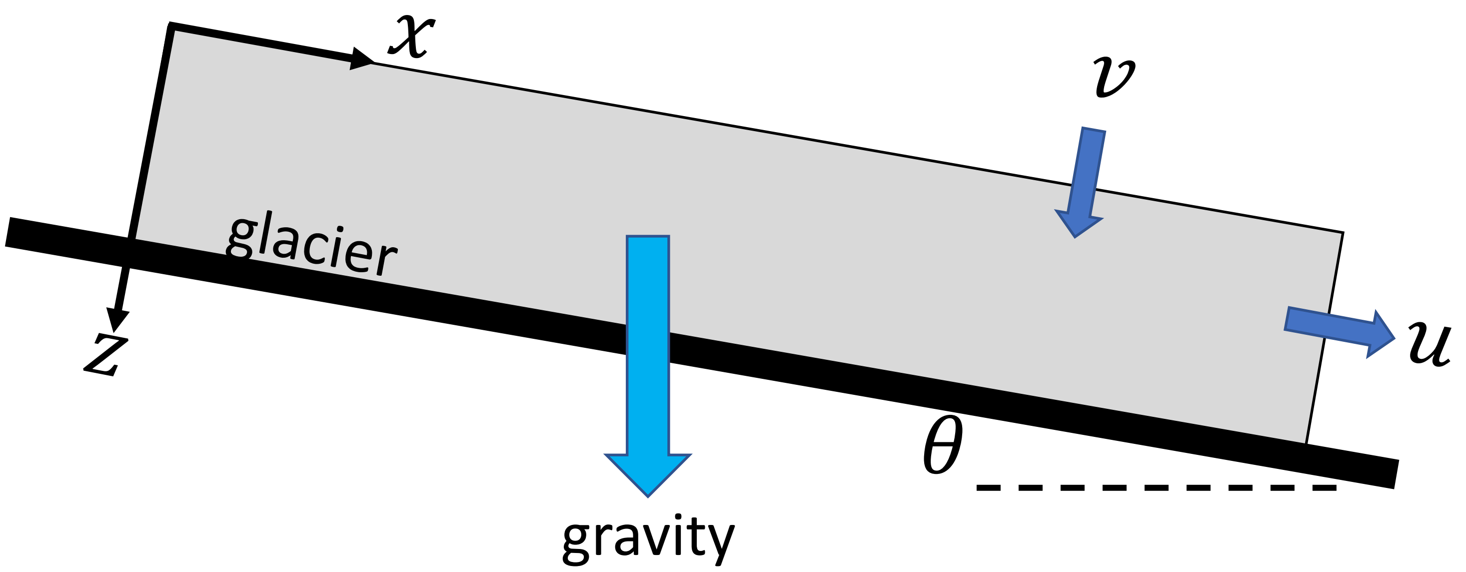


$$\sigma_x = -\rho g z \cos\theta \pm 2\sqrt{\tau^2 - (\rho g z \sin\theta)^2}$$

Why two solutions?

As we will see in few slides
its because the glacier can be
either extending or compressing.

Variation of Velocity with Position

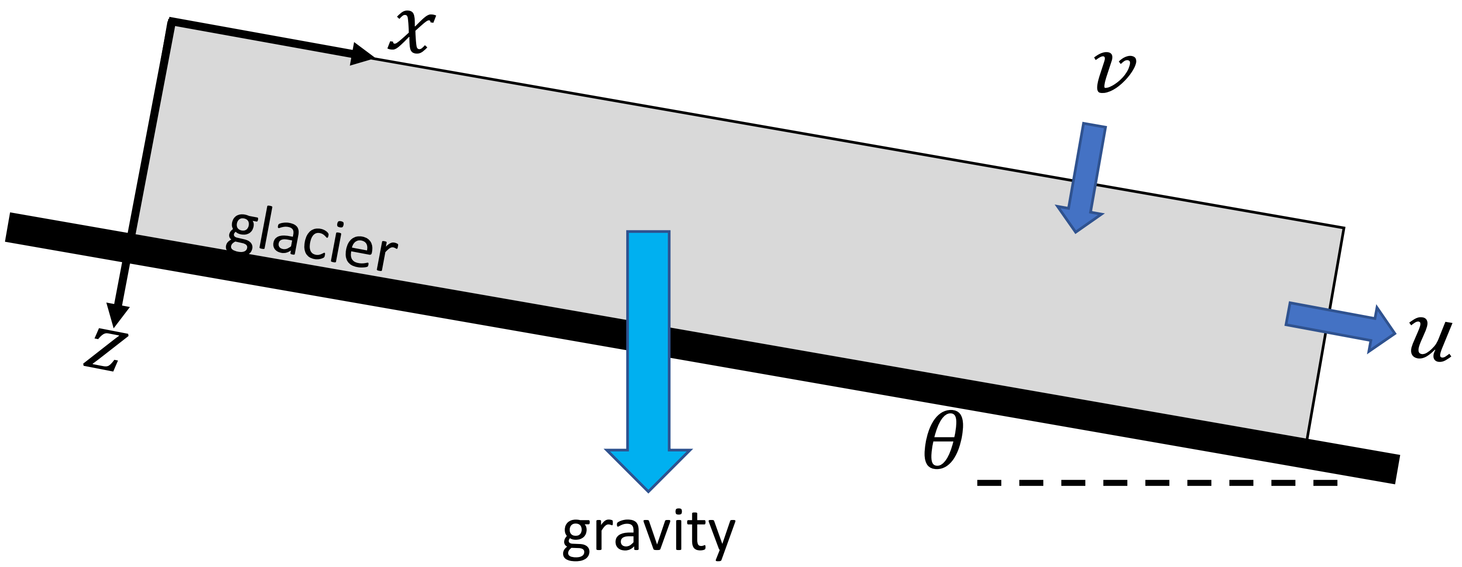


flow law

$$\dot{\varepsilon}_x = \frac{du}{dx} = A\tau^2 \tfrac{1}{2}(\sigma_x - \sigma_z)$$

$$\dot{\varepsilon}_{xy} = \tfrac{1}{2}\left(\frac{du}{dz} + \frac{dv}{dx}\right) = A\tau^2 \tau_{xy}$$

independent of x



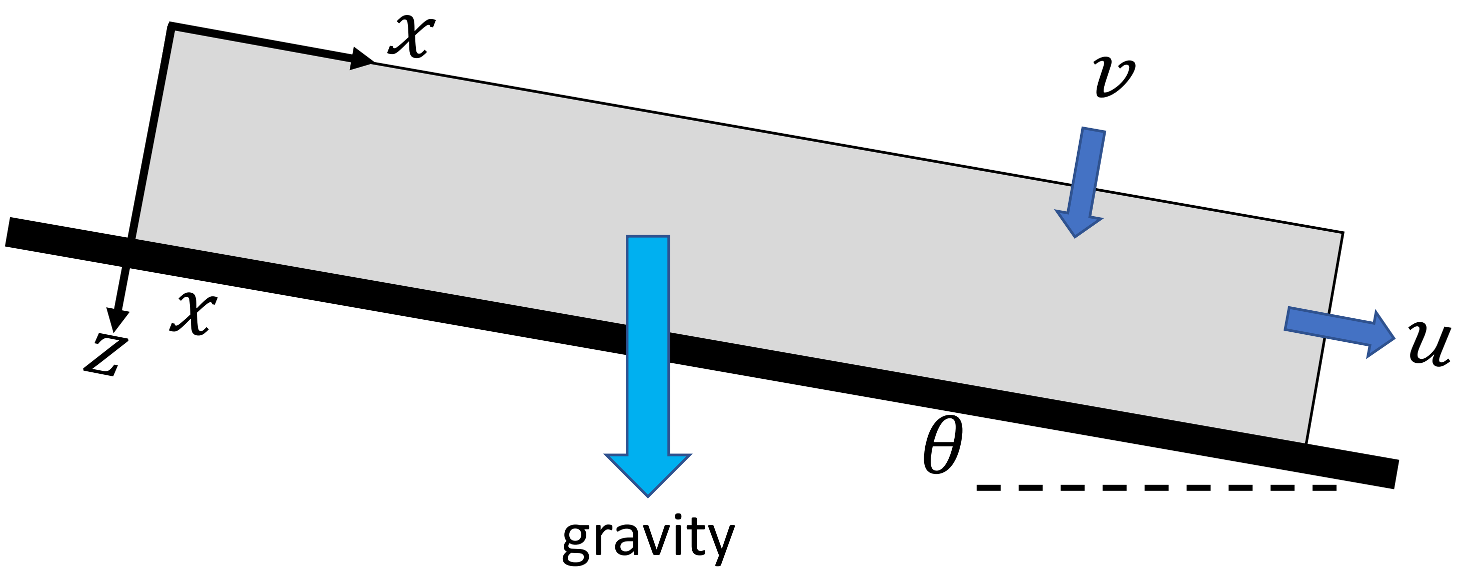
flow law

$$\dot{\varepsilon}_x = \frac{du}{dx} = A\tau^2 \tfrac{1}{2}(\sigma_x - \sigma_z)$$

$$\dot{\varepsilon}_{xy} = \tfrac{1}{2}\left(\frac{du}{dz} + \frac{dv}{dx}\right) = A\tau^2 \tau_{xy}$$

independent of x

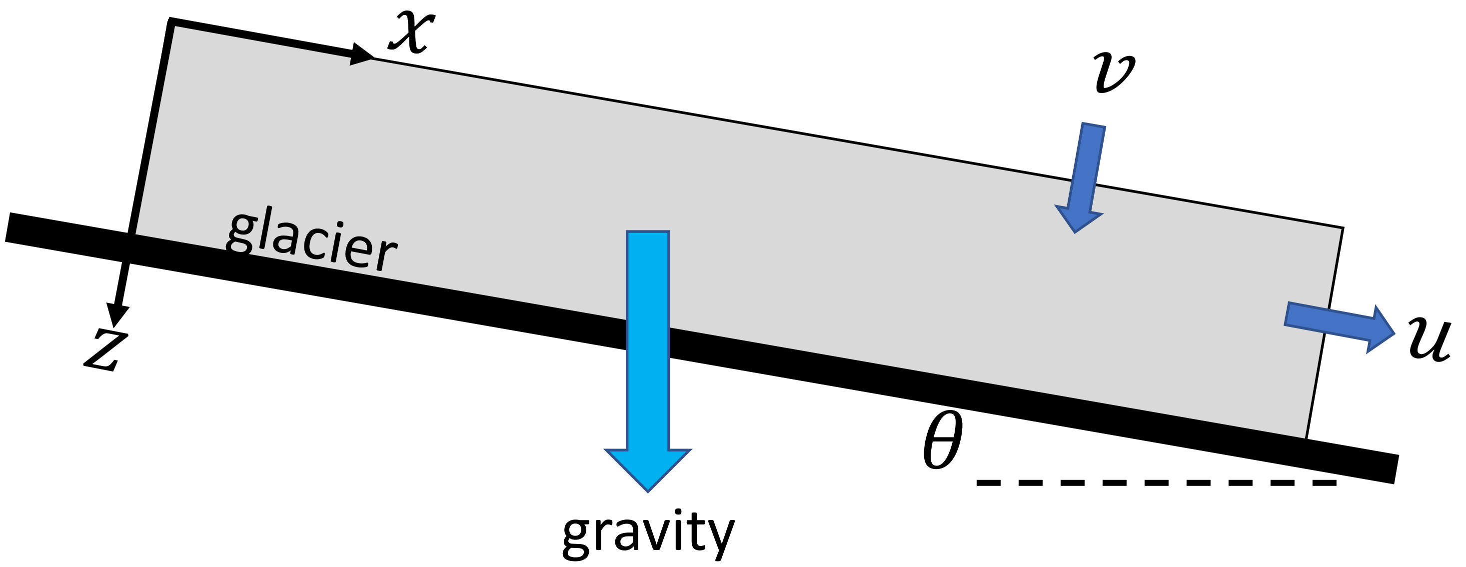
independent of x, too



flow law

$$\dot{\varepsilon}_x = \frac{du}{dx} = A\tau^2 ½(\sigma_x - \sigma_z) \quad \overset{\frac{d}{dx}}{\text{implies}} \quad \frac{d^2u}{dx^2} = 0$$

$$\dot{\varepsilon}_{xy} = ½\left(\frac{du}{dz} + \frac{dv}{dx}\right) = A\tau^2\tau_{xy} \quad \overset{\frac{d}{dx}}{\text{implies}} \quad \frac{d^2u}{dxdx} + \frac{d^2u}{dx^2} = 0$$



flow law

$$\dot{\varepsilon}_x = \frac{du}{dx} = A\tau^2 ½(\sigma_x - \sigma_z)$$

$\frac{d}{dx}$ implies $\frac{d^2u}{dx^2} = 0$

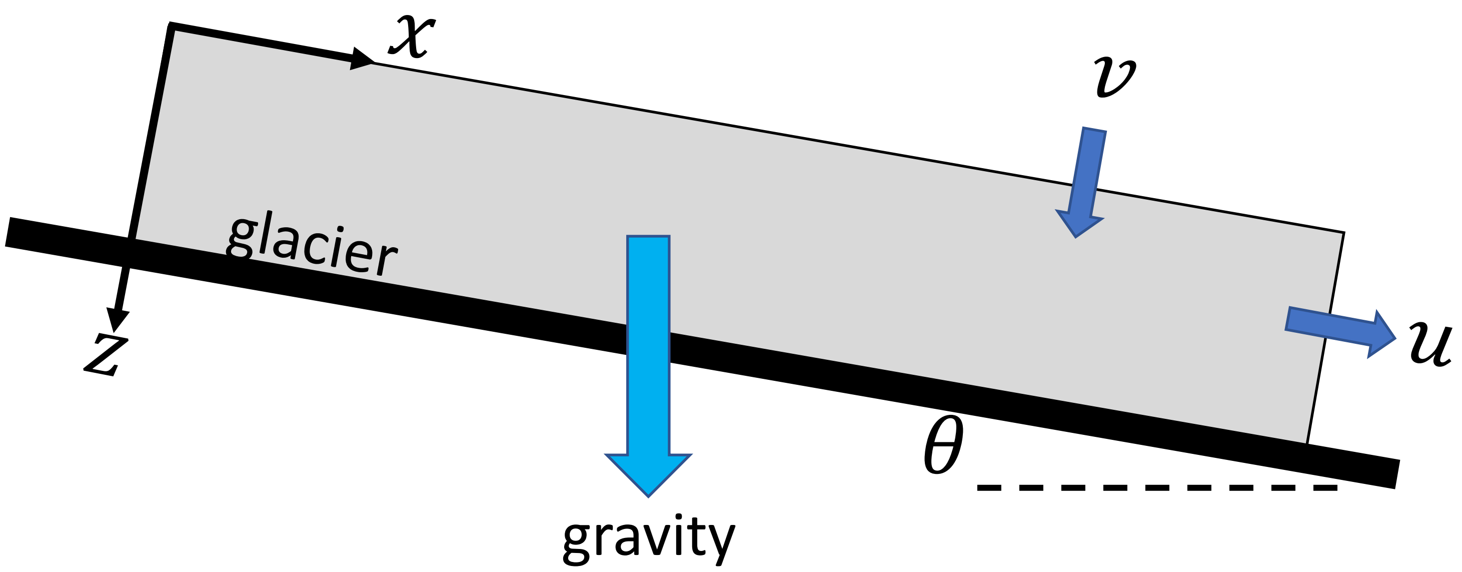
$$\dot{\varepsilon}_{xy} = ½\left(\frac{du}{dz} + \frac{dv}{dx}\right) = A\tau^2\tau_{xy}$$

$\frac{d}{dx}$ implies $\frac{d^2u}{dxdz} + \frac{d^2v}{dx^2} = 0$

implies

$$u = u_0 \pm rx + f(z)$$
$$v = \pm r(H - z)$$

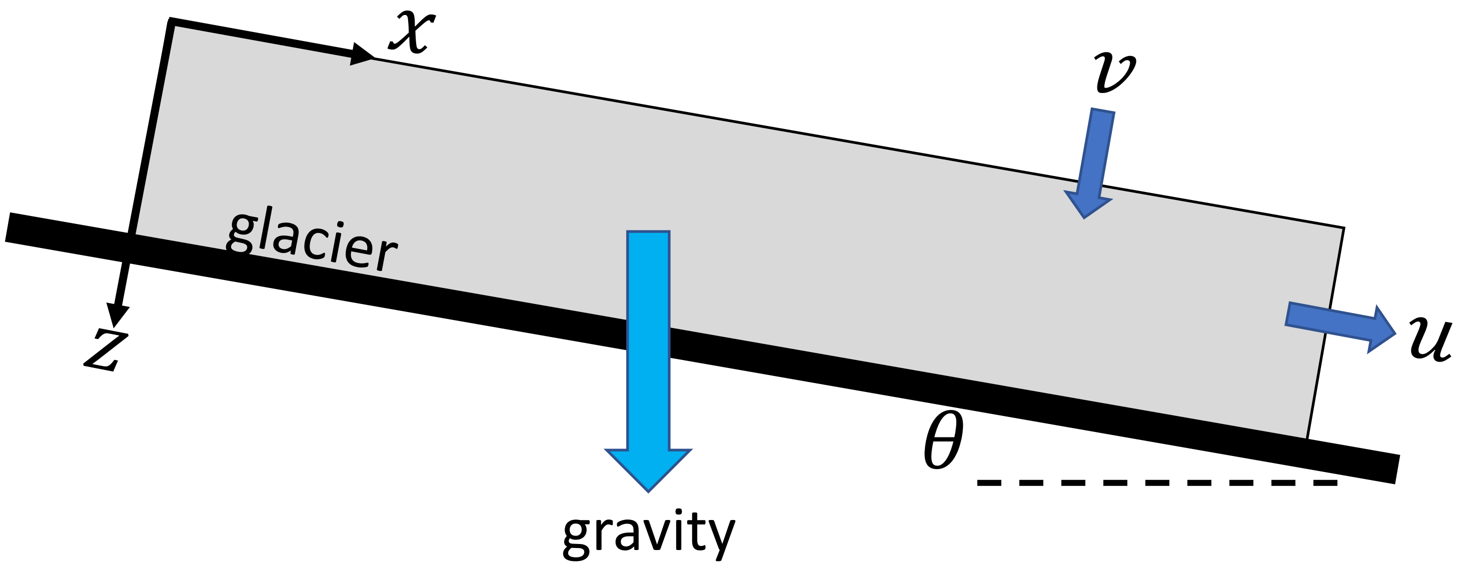
with $f(z = 0) = 0$



still need to figure out f

$$u = u_0 \pm rx + f(z)$$
$$v = \pm r(H - z)$$

with
$f(z = 0) = 0$

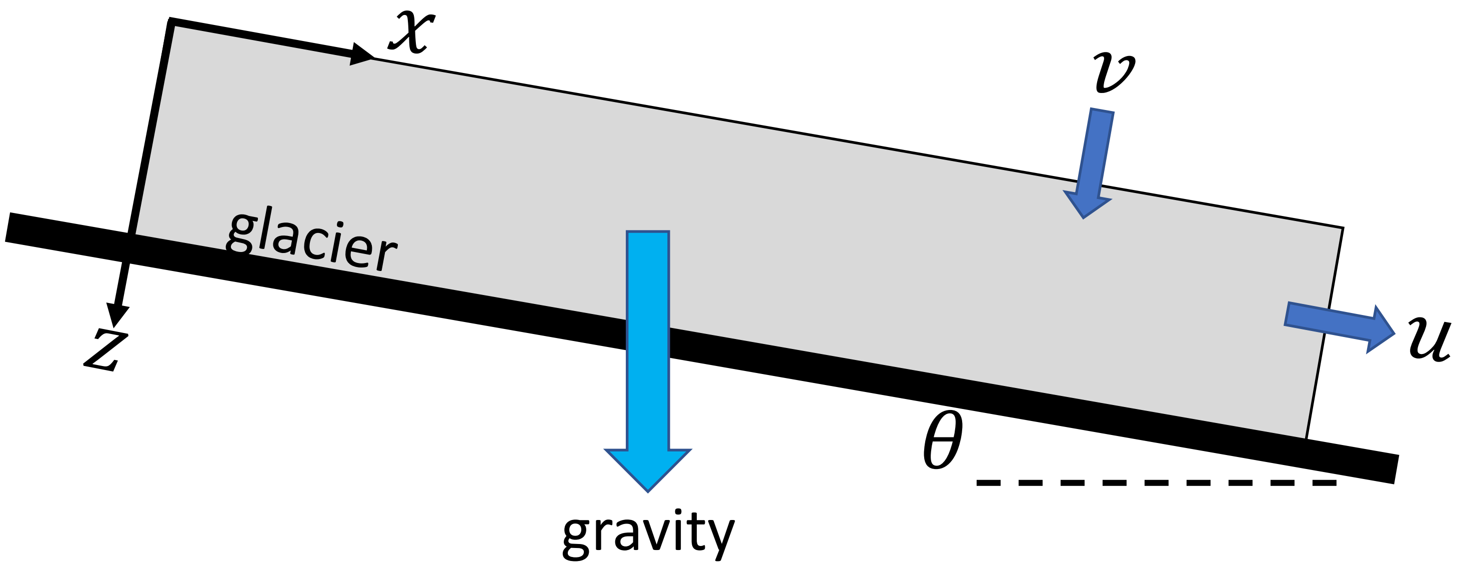


$\pm r$:

$+r$: glacier extending and thinning

$-r$: glacier compressing and thickening

$$u = u_0 \pm rx + f(z)$$
$$v = \pm r(H - z)$$

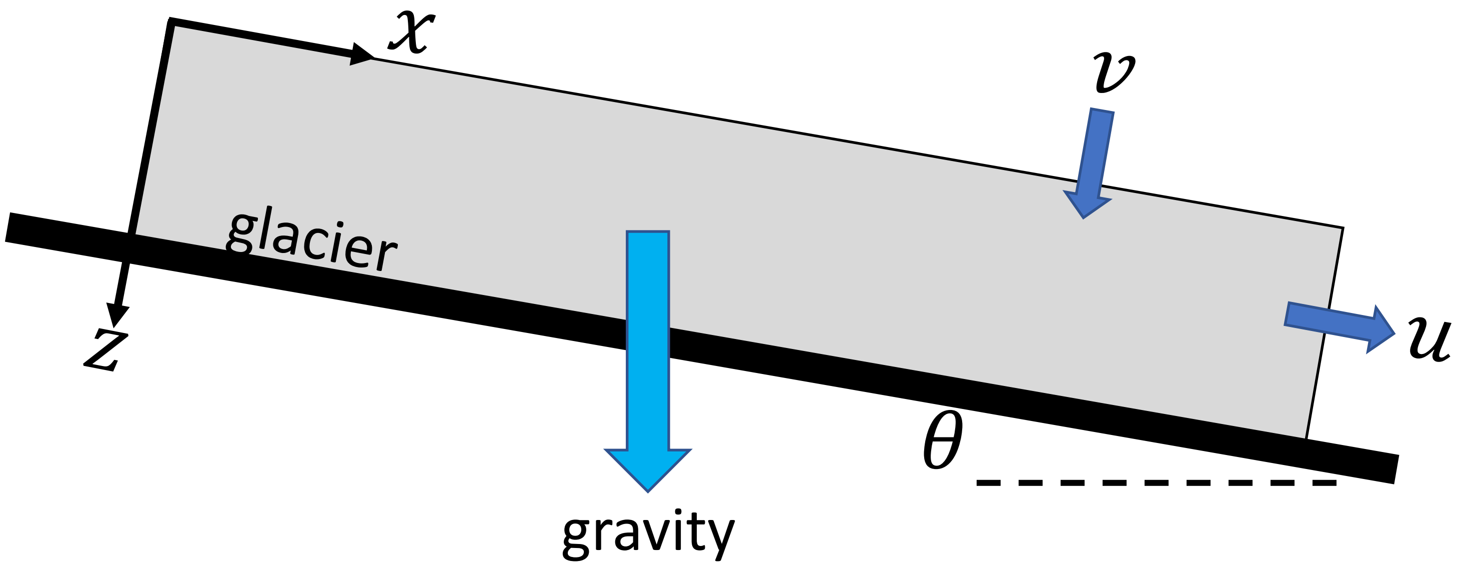


flow law $$\frac{du}{dx} = ½A\tau^2(\sigma_x - \sigma_z)$$

$$u = u_0 \pm rx + f(z)$$

$$\sigma_x = \sigma_z + \frac{2}{A\tau^2}\frac{du}{dx}$$

$$\sigma_x = \sigma_z \pm r\frac{2}{A\tau^2}$$

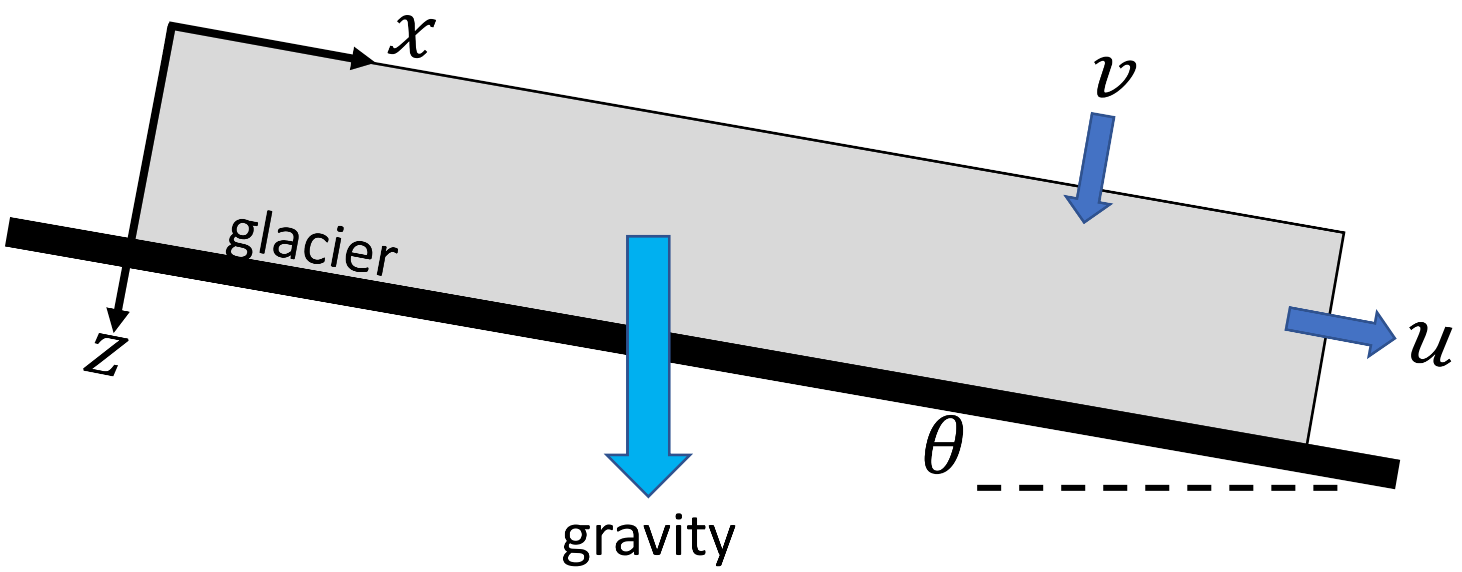


$$u = u_0 \pm rx + f(z)$$

compare with

$$\sigma_x = \sigma_z \pm r\frac{2}{A\tau^2}$$

$$\sigma_x = \sigma_z \pm 2\sqrt{\tau^2 - \tau_{xy}^2}$$



$$u = u_0 \pm rx + f(z)$$

so two solutions of quadratic equation correspond to choice of $\pm r$

$$\sigma_x = \sigma_z \pm r\frac{2}{A\tau^2}$$

$$\sigma_x = \sigma_z \pm 2\sqrt{\tau^2 - \tau_{xy}^2}$$

so choosing between the two roots

is the same a choosing the sign of r

$$u = u_0 \pm rx + f(z)$$

Figuring out f and τ

Step 1 of determining τ and f: relate f to τ

$$\dot{\varepsilon}_{xy} = ½\left(\frac{du}{dz} + \frac{dv}{dx}\right) = A\tau^2\tau_{xy}$$

$$u = u_0 + rx + f(z)$$

$$v = r(H - z)$$

$$\tau_{xz} = -\sigma g z \sin\theta$$

combine

$$\frac{df}{dz} = -(2A\sigma g \sin\theta) z\tau^2$$

Step 2 of determining τ and f: another way to relate f to τ

substitute

$$u = u_0 + rx + f(z)$$
$$v = r(H - z)$$
$$\dot{\varepsilon} = A\tau^3$$

into the state of strain rate law

$$2\dot{\varepsilon}^2 = \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dz}\right)^2 + ½\left(\frac{du}{dz} + \frac{dv}{dx}\right)^2$$

to get

$$2A^2\tau^6 = r^2 + r^2 + ½\left(\frac{df}{dz} + 0\right)^2$$

$$2A^2\tau^6 = 2r^2 + ½\left(\frac{df}{dz}\right)^2$$

Step 3 of determining τ and f: eliminate f to get equation for τ

$$2A^2\tau^6 = 2r^2 + ½\left(\frac{df}{dz}\right)^2$$

$$\frac{df}{dz} = -(2A\rho g \sin\theta) z\tau^2$$

cubiic equation in τ^2

$$2(\tau^2)^3 - ½(2\rho g z \sin\theta)^2(\tau^2)^2 - 2\left(\frac{r}{A}\right)^2 = 0$$

$$(\tau^2)^3 - (\rho g z \sin\theta)^2(\tau^2)^2 - \left(\frac{r}{A}\right)^2 = 0$$

cubic equation for τ^2

note τ^2 depends on z

cubic has 3 roots

but τ^2 must be positive

Descartes rule of signs

The number of positive roots is at most the number of sign changes in the sequence of polynomial's coefficients (omitting the zero coefficients). if the number of sign changes is one, then there are exactly one positive roots

$$+(\tau^2)^3 - (\rho g z \sin\theta)^2(\tau^2)^2 - \left(\frac{r}{A}\right)^2 = 0$$

1 sign change

so only 1 positive solution

Step 4 of determining τ and f: solve for f

solve $\frac{df}{dz}$ for f

$$\frac{df}{dz} = -(2A\sigma g \sin\theta) z \tau^2 \quad \text{with} \quad f(z = 0)$$

Putting it together

What must the extension rate be to get crevasses?

How deep into the glacier will the crevasses extend?

equation for τ^2

$$(\tau^2)^3 - (\rho g z \sin\theta)^2 (\tau^2)^2 - \left(\frac{r}{A}\right)^2 = 0$$

$z = 0$

$$(\tau^2)^3 - (\rho g z \sin\theta)^2 (\tau^2)^2 - \left(\frac{r}{A}\right)^2 = 0$$

$$\tau^2 = \left(\frac{r}{A}\right)^{2/3}$$

$large\ z$

$$(\tau^2)^3 - (\rho g z \sin\theta)^2 (\tau^2)^2 - \left(\frac{r}{A}\right)^2 = 0$$

$$\tau^2 = (\rho g z \sin\theta)^2$$

equation for σ_x

$$\sigma_x = -\rho g z \cos\theta \pm 2\sqrt{\tau^2 - (\rho g z \sin\theta)^2}$$

$z = 0$

$$\tau^2 = \left(\frac{r}{A}\right)^{2/3} \qquad \sigma_x = \pm 2\sqrt{\left(\frac{r}{A}\right)^{2/3}}$$

$large\ z$

$$\tau^2 = (\rho g z \sin\theta)^2 \qquad \sigma_x = -\rho g z \cos\theta$$

equation for σ_x

$$\sigma_x = -\rho g z \cos\theta \pm 2\sqrt{\tau^2 - (\rho g z \sin\theta)^2}$$

$z = 0$

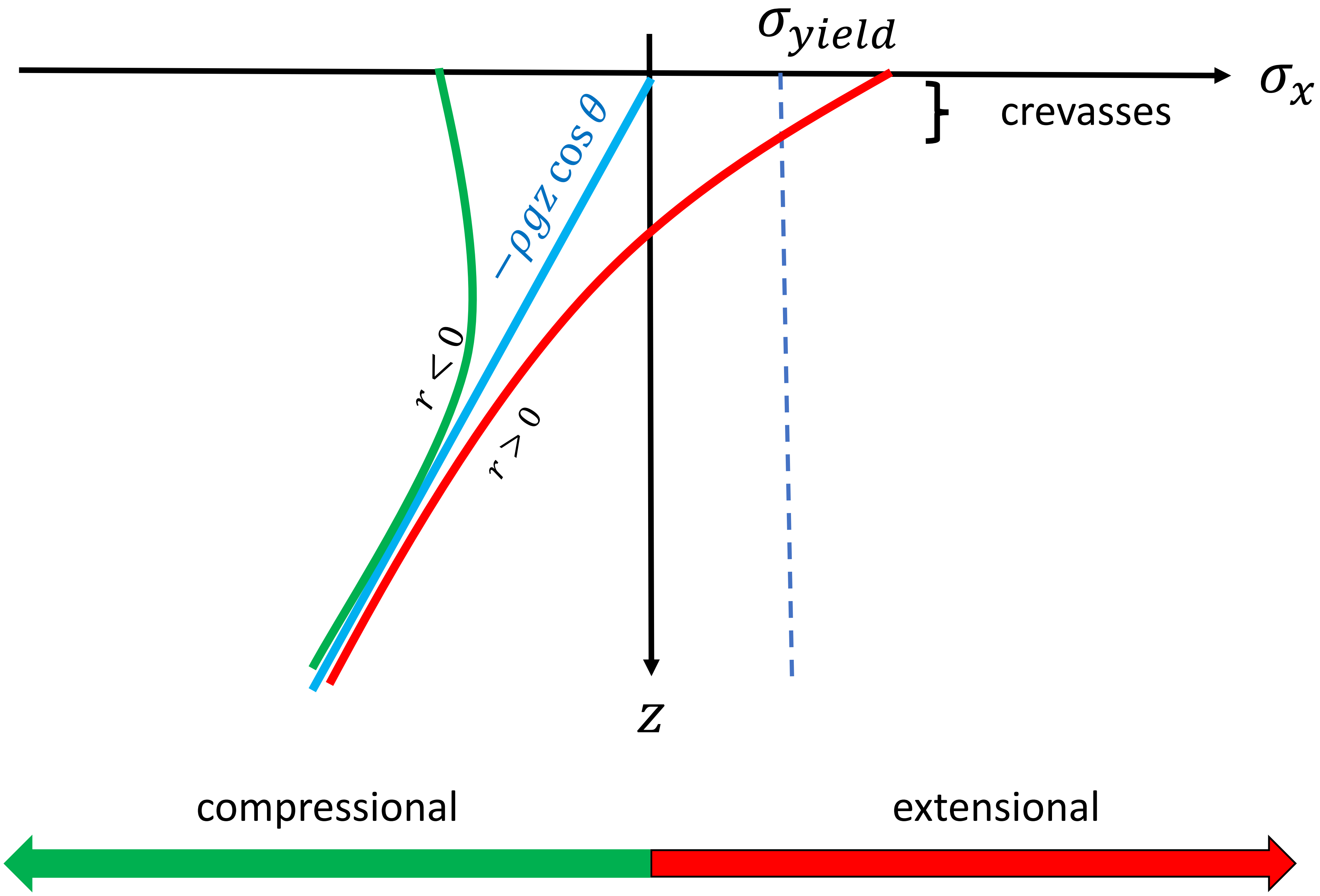
$$\tau^2 = \left(\frac{r}{A}\right)^{2/3} \qquad \sigma_x = \pm 2\sqrt{\left(\frac{r}{A}\right)^{2/3}}$$

To get crevasses, you must chose the + solution and σ_x must exceed the yield stress

$large\ z$

$$\tau^2 = \rho g z \sin\theta \qquad \sigma_x = -\rho g z \cos\theta$$

Irrespective of the choice of the $\pm$, the stress becomes compressive at very deep depth. So the + solution only has extensional stress in the upper part of the glacier. You must solve for τ(z) to determine how deep extension extends.



σ_{yield}
σ_x
crevasses
$-\rho g z \cos\theta$
$r < 0$
$r > 0$
z
compressional
extensional

Indian climber spent three days stricken on the world's 10th tallest peak (Image: Twitter/@anuragmaloo)

Climber rescued alive after spending three DAYS inside skyscraper-sized crevasse