Contents

Introduction xv

1. Describing Inverse Problems

1.1 Formulating Inverse Problems 1
1.1.1 Implicit Linear Form 2
1.1.2 Explicit Form 2
1.1.3 Explicit Linear Form 3
1.2 The Linear Inverse Problem 3
1.3 Examples of Formulating Inverse Problems 4
1.3.1 Example 1: Fitting a Straight Line 4
1.3.2 Example 2: Fitting a Parabola 5
1.3.3 Example 3: Acoustic Tomography 6
1.3.4 Example 4: X-ray Imaging 7
1.3.5 Example 5: Spectral Curve Fitting 9
1.3.6 Example 6: Factor Analysis 10
1.4 Solutions to Inverse Problems 11
1.4.1 Estimates of Model Parameters 11
1.4.2 Bounding Values 12
1.4.3 Probability Density Functions 12
1.4.4 Sets of Realizations of Model Parameters 13
1.4.5 Weighted Averages of Model Parameters 13
1.5 Problems 13

2. Some Comments on Probability Theory

2.1 Noise and Random Variables 15
2.2 Correlated Data 19
2.3 Functions of Random Variables 21
2.4 Gaussian Probability Density Functions 26
2.5 Testing the Assumption of Gaussian Statistics 29
2.6 Conditional Probability Density Functions 30
2.7 Confidence Intervals 33
2.8 Computing Realizations of Random Variables 34
2.9 Problems 37

3. Solution of the Linear, Gaussian Inverse Problem, Viewpoint 1: The Length Method

3.1 The Lengths of Estimates 39
3.2 Measures of Length 39
3.3 Least Squares for a Straight Line 43
3.4 The Least Squares Solution of the Linear Inverse Problem 44
3.5 Some Examples 46
3.5.1 The Straight Line Problem 46
3.5.2 Fitting a Parabola 47
3.5.3 Fitting a Plane Surface 48
3.6 The Existence of the Least Squares Solution 49
3.6.1 Underdetermined Problems 51
3.6.2 Even-Determined Problems 52
3.6.3 Overdetermined Problems 52
3.7 The Purely Underdetermined Problem 52
3.8 Mixed-Determined Problems 54
3.9 Weighted Measures of Length as a Type of *A Priori* Information 56
3.9.1 Weighted Least Squares 58
3.9.2 Weighted Minimum Length 58
3.9.3 Weighted Damped Least Squares 58
3.10 Other Types of *A Priori* Information 60
3.10.1 Example: Constrained Fitting of a Straight Line 62
3.11 The Variance of the Model Parameter Estimates 63
3.12 Variance and Prediction Error of the Least Squares Solution 64
3.13 Problems 67

4. Solution of the Linear, Gaussian Inverse Problem, Viewpoint 2: Generalized Inverses

4.1 Solutions Versus Operators 69
4.2 The Data Resolution Matrix 69
4.3 The Model Resolution Matrix 72
4.4 The Unit Covariance Matrix 72
4.5 Resolution and Covariance of Some Generalized Inverses 74
4.5.1 Least Squares 74
4.5.2 Minimum Length 75
4.6 Measures of Goodness of Resolution and Covariance 75
4.7 Generalized Inverses with Good Resolution and Covariance 76
4.7.1 Overdetermined Case 76
4.7.2 Underdetermined Case 77
4.7.3 The General Case with Dirichlet Spread Functions 77
4.8 Sidelobes and the Backus-Gilbert Spread Function 78
4.9 The Backus-Gilbert Generalized Inverse for the Underdetermined Problem 79
4.10 Including the Covariance Size 83

4.11 The Trade-off of Resolution and Variance 84
4.12 Techniques for Computing Resolution 86
4.13 Problems 88

5. Solution of the Linear, Gaussian Inverse Problem, Viewpoint 3: Maximum Likelihood Methods

5.1 The Mean of a Group of Measurements 89
5.2 Maximum Likelihood Applied to Inverse Problem 92
5.2.1 The Simplest Case 92
5.2.2 *A Priori* Distributions 92
5.2.3 Maximum Likelihood for an Exact Theory 97
5.2.4 Inexact Theories 100
5.2.5 The Simple Gaussian Case with a Linear Theory 102
5.2.6 The General Linear, Gaussian Case 104
5.2.7 Exact Data and Theory 107
5.2.8 Infinitely Inexact Data and Theory 108
5.2.9 No *A Priori* Knowledge of the Model Parameters 108
5.3 Relative Entropy as a Guiding Principle 108
5.4 Equivalence of the Three Viewpoints 110
5.5 The *F*-Test of Error Improvement Significance 111
5.6 Problems 113

6. Nonuniqueness and Localized Averages

6.1 Null Vectors and Nonuniqueness 115
6.2 Null Vectors of a Simple Inverse Problem 116
6.3 Localized Averages of Model Parameters 117
6.4 Relationship to the Resolution Matrix 117
6.5 Averages Versus Estimates 118
6.6 Nonunique Averaging Vectors and *A Priori* Information 119
6.7 Problems 121

7. Applications of Vector Spaces

7.1 Model and Data Spaces 123
7.2 Householder Transformations 124
7.3 Designing Householder Transformations 127
7.4 Transformations That Do Not Preserve Length 129
7.5 The Solution of the Mixed-Determined Problem 130
7.6 Singular-Value Decomposition and the Natural Generalized Inverse 132
7.7 Derivation of the Singular-Value Decomposition 138
7.8 Simplifying Linear Equality and Inequality Constraints 138
7.8.1 Linear Equality Constraints 139
7.8.2 Linear Inequality Constraints 139
7.9 Inequality Constraints 140
7.10 Problems 147

8. Linear Inverse Problems and Non-Gaussian Statistics

8.1 L_1 Norms and Exponential Probability Density Functions 149
8.2 Maximum Likelihood Estimate of the Mean of an Exponential Probability Density Function 151
8.3 The General Linear Problem 153
8.4 Solving L_1 Norm Problems 153
8.5 The L_∞ Norm 158
8.6 Problems 160

9. Nonlinear Inverse Problems

9.1 Parameterizations 163
9.2 Linearizing Transformations 165
9.3 Error and Likelihood in Nonlinear Inverse Problems 166
9.4 The Grid Search 167
9.5 The Monte Carlo Search 170
9.6 Newton's Method 171
9.7 The Implicit Nonlinear Inverse Problem with Gaussian Data 175
9.8 Gradient Method 180
9.9 Simulated Annealing 181
9.10 Choosing the Null Distribution for Inexact Non-Gaussian Nonlinear Theories 184
9.11 Bootstrap Confidence Intervals 185
9.12 Problems 186

10. Factor Analysis

10.1 The Factor Analysis Problem 189
10.2 Normalization and Physicality Constraints 194
10.3 Q-Mode and R-Mode Factor Analysis 199
10.4 Empirical Orthogonal Function Analysis 199
10.5 Problems 204

11. Continuous Inverse Theory and Tomography

11.1 The Backus-Gilbert Inverse Problem 207
11.2 Resolution and Variance Trade-Off 209
11.3 Approximating Continuous Inverse Problems as Discrete Problems 209
11.4 Tomography and Continuous Inverse Theory 211
11.5 Tomography and the Radon Transform 212
11.6 The Fourier Slice Theorem 213
11.7 Correspondence Between Matrices and Linear Operators 214
11.8 The Fréchet Derivative 218
11.9 The Fréchet Derivative of Error 218
11.10 Backprojection 219
11.11 Fréchet Derivatives Involving a Differential Equation 222
11.12 Problems 227

12. Sample Inverse Problems

12.1 An Image Enhancement Problem 231
12.2 Digital Filter Design 234
12.3 Adjustment of Crossover Errors 236
12.4 An Acoustic Tomography Problem 240
12.5 One-Dimensional Temperature Distribution 241
12.6 L_1, L_2, and L_∞ Fitting of a Straight Line 245
12.7 Finding the Mean of a Set of Unit Vectors 246
12.8 Gaussian and Lorentzian Curve Fitting 250
12.9 Earthquake Location 252
12.10 Vibrational Problems 256
12.11 Problems 259

13. Applications of Inverse Theory to Solid Earth Geophysics

13.1 Earthquake Location and Determination of the Velocity Structure of the Earth from Travel Time Data 261
13.2 Moment Tensors of Earthquakes 264
13.3 Waveform "Tomography" 265
13.4 Velocity Structure from Free Oscillations and Seismic Surface Waves 267
13.5 Seismic Attenuation 269
13.6 Signal Correlation 270
13.7 Tectonic Plate Motions 271
13.8 Gravity and Geomagnetism 271
13.9 Electromagnetic Induction and the Magnetotelluric Method 273

14. Appendices

14.1 Implementing Constraints with Lagrange multipliers 277
14.2 L_2 Inverse Theory with Complex Quantities 278

Index 281