Energy in a wave on a string.

Basic Principles!

Kinetic energy density $= \frac{1}{2}\rho \dot{y}^2 = k$

potential energy density = force × stretched distance of string $= \frac{T}{2}\left(\frac{dy}{dx}\right)^2$

$$ds = (dx^2 + dy^2)^{1/2}$$

stretched distance =

$$ds - dx = dl$$

$$\rho v^2 = T$$

$$dl = \left[\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} - 1\right] dx$$

$$\approx \left(1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 - 1\right)dx = \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

for the wave we have $y = A \sin \frac{2\pi}{\lambda}(x - vt)$

$$\dot{y} = A\frac{2\pi v}{\lambda} \cos \frac{2\pi}{\lambda}(x - vt)$$

$$y_x = A\frac{2\pi}{\lambda} \cos \frac{2\pi}{\lambda}(x - vt)$$

at $t=0$ E in one wavelength λ =

$$E = \int_0^\lambda \left[\frac{1}{2}\rho \frac{4\pi^2 v^2}{\lambda^2} A^2 \cos^2 \frac{2\pi}{\lambda}(x - vt) + \frac{1}{2} v^2 \rho A^2 \frac{4\pi^2}{\lambda^2} \cos^2\left(\frac{2\pi}{\lambda}(x - vt)\right)\right] dx$$

$$= \int_0^\lambda \left(\frac{2\pi^2 A^2 \rho v^2}{\lambda^2} + \frac{2\pi^2 A^2 v^2 \rho}{\lambda^2}\right) \cos^2\left(\frac{2\pi}{\lambda}(x - vt)\right) dx$$

note Pot E = Kin E

$$= \frac{4\pi^2 A^2 \rho v^2}{\lambda^2} \frac{\lambda}{2\pi} \int_0^{2\pi} \cos^2 y \, dy \quad ; \quad \frac{\lambda}{2\pi} y = x$$

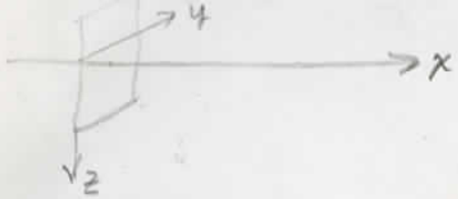
$$\text{"}\pi$$

$$= \frac{2\pi^2 A^2 \rho v^2}{\lambda} = \frac{2\pi^2 T A^2}{\lambda}$$

Energy flux = $\sigma_{ij} \frac{\partial u_i}{\partial t}$ (Bullen)

Energy flux / unit area in a plane P wave

let p wave move to right at velocity α. Then



$$\vec{u} = A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sin(kx - \omega t)$$

now kinetic energy density $= K = \frac{1}{2} \rho \dot{u}^2 = \frac{1}{2} \rho A^2 \omega^2 \cos^2(kx - \omega t)$

now potential energy density = strain energy density $= \frac{1}{2} e_{ij} \sigma_{ij}$ but because of hook's law $\sigma_{ij} = C_{ijpq} e_{pq}$ $\quad \phi = e_{ij} C_{ijpq} e_{pq}$

strain $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$C_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})$

so $\phi = \frac{1}{8}(u_{i,j} + u_{j,i})(\lambda \delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}))(u_{p,q} + u_{q,p})$

only $u_{i,j}$ that is non-zero is $u_{1,1}$ so $i=1,\ j=1$
$p=1,\ q=1$

and

$$\phi = \frac{1}{8}(u_{1,1} + u_{1,1})(\lambda + 2\mu)(u_{1,1} + u_{1,1})$$

$$= \frac{1}{2}(\lambda + 2\mu)\, u_{1,1}^2 = \frac{1}{2}(\lambda + 2\mu) A^2 k^2 \cos^2(kx - \omega t)$$

now $v^2 = \frac{\lambda + 2\mu}{\rho} = \frac{\omega^2}{k^2}$ $\quad \lambda + 2\mu = \rho \frac{\omega^2}{k^2}$

$$\phi = \frac{1}{2} \rho \omega^2 A^2 \cos^2(kx - \omega t)$$

$$E = \int_0^\lambda \rho \omega^2 A^2 \cos^2\left(\frac{2\pi}{\lambda} x\right) = \rho \omega^2 A^2 \frac{\lambda}{2\pi} \pi =$$

$$= \frac{\pi \rho \omega^2 A^2}{k} = \frac{2\pi^2 \rho \alpha^2 A^2}{\lambda} = \pi \omega \rho \alpha A^2$$

$\frac{\omega}{k} = \alpha \quad = \pi \rho \alpha A^2$

$\frac{\omega}{2\pi} = f$

$\lambda f = \alpha$

$\alpha = \omega / k$

note for energy balance perpendicular

to surface multiply by $\cos i$ or $\cos j$