Yash asked me if a particular surface integral could be written as a line integral. Here is problem and my solution:

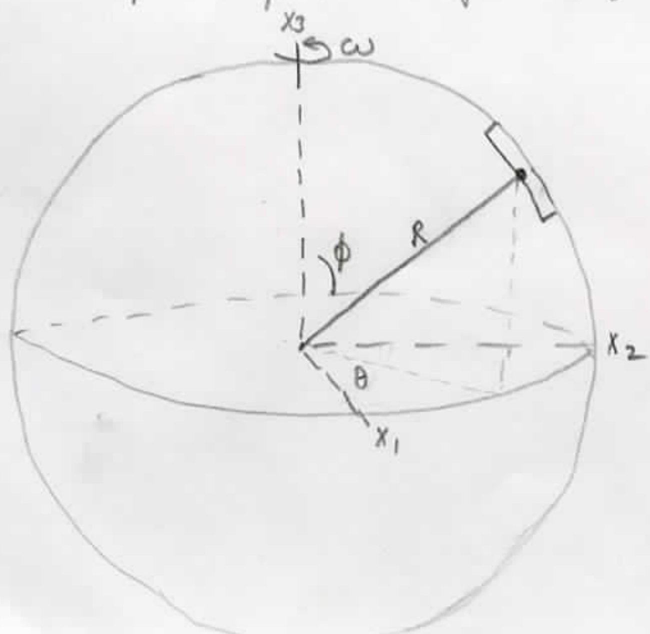
problem: given spherical earth with rigid continent rotating at velocity ω with respect to pole of rotation. Drag on the bottom of the plate is porportional to velocity, therefore the net torque M on the plate is

$$\underline{M} = \iint_{\text{Plate Area}} (\underline{R} \times \underline{T}_{SHEAR})\, da$$

where $\underline{T}_{shear}$ is the shear traction. we assume $\underline{T}_{shear} = \alpha V$. On a rigid body $\underline{V} = \underline{\omega} \times \underline{r}$ so the torque is;

$$\underline{M} = \iint_{\text{Plate area}} \alpha\, (r \times (\omega \times r))\, da$$

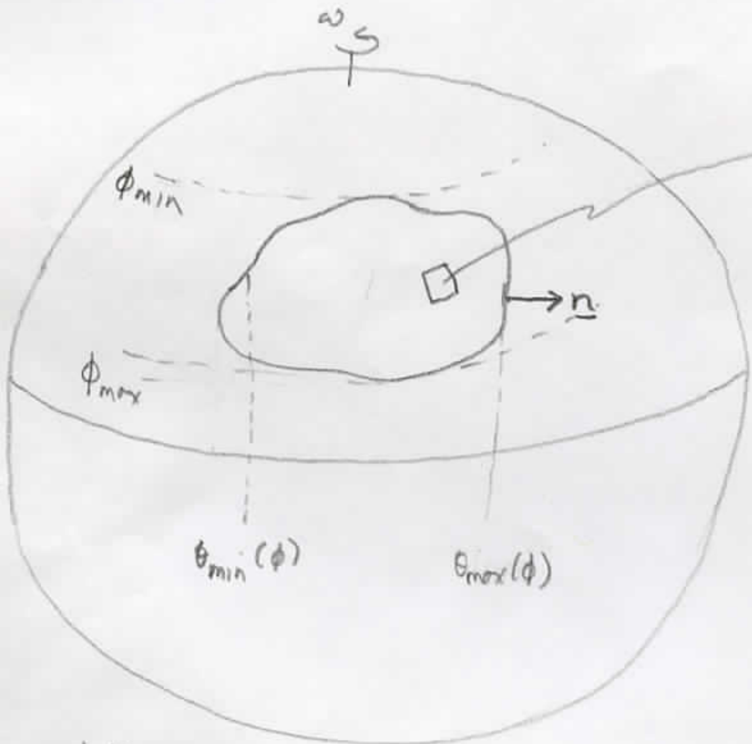
can we WRITE This as a line integral? solution;
now consider the following plate geometry



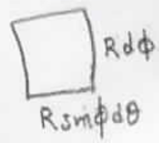
by this geometry it's easy to write down the components of the torque. note $M_1 = M_2 = 0$

$$M_3 = \iint_{\text{Plate area}} \alpha R^2 \omega \sin^2\phi \, da$$

now let's look at a plate close up



small area element da



$da = R^2 \sin\phi \, d\theta d\phi$

$ds^2 = R^2 d\phi^2 + R^2 \sin^2\phi \, d\theta^2$

$\underline{n} \equiv$ unit vector on surface of sphere, perpendicular to boundary of plate.

in this notation

$$M_3 = \int_{\phi_{min}}^{\phi_{max}} R d\phi \int_{\theta_{min}(\phi)}^{\theta_{max}(\phi)} R \sin\phi \, d\theta \quad \alpha R^2 \omega \sin^2\phi \, da$$

now consider the following form of Green's thm, good for any suitably differentiable vector function G

$$\iint_{\text{Region}} \nabla \cdot \underline{G} \, da = \oint_{\text{Boundary of Region}} \underline{G} \cdot \underline{n} \, ds$$

for a vector lying in the surface of a sphere, the appropriate rule for the divergence is :

$$\nabla \cdot G = \frac{1}{R\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, G_\phi) + \frac{1}{R\sin\phi}\frac{\partial G_\theta}{\partial\theta} .$$

now we would like to pick some function G such that the Left hand side of Green's Theorum becomes the same as the expression for M_3. A perfectly good choice is

$$G_\phi = 0 \qquad G_\theta = \alpha R^3 \omega \theta \sin^5\phi$$

which gives

$$\oint \underline{G}\cdot n \, ds = \iint_{\text{AREA of Plate}} \nabla\cdot G \, da = \iint_{\text{AREA}} \alpha R^2 \omega \sin^2\phi \, da = M_3$$

now notice ~~that~~ the vector V has components

$$V_\phi = 0 \qquad V_\theta = R\omega\sin\phi \quad .$$ so G is portional to V

$$\underline{G} = \alpha R^2 \theta \sin^2\phi \, \underline{V}$$

and the torque becomes

$$\alpha R^2 \oint \theta \sin^2\phi \, \underline{V}\cdot\underline{N} \, ds = M_3$$