Modes of Vibration of Volcano tube. W/ S McNutt
Fall 80

MRN 016

acoustic equation $\quad \omega^2 p + \alpha^2 \nabla^2 p + \lambda \nabla p \cdot \nabla \frac{1}{\rho} = 0$

in one dimension $\quad \alpha^2 \frac{d^2 p}{dz^2} + \lambda \frac{d\frac{1}{\rho}}{dz} \frac{dp}{dz} + \omega^2 p = 0$

$$\alpha^2 = \frac{\lambda}{\rho} \qquad \frac{d}{dz} \rho^{-1} = -\rho^{-2} \frac{d\rho}{dz}$$

$$\frac{\lambda}{\rho} \frac{d^2 p}{dz^2} - \frac{\lambda}{\rho^2} \frac{d\rho}{dz} \frac{dp}{dz} + \omega^2 p = 0$$

$$\lambda \rho \frac{d^2 p}{dz^2} - \lambda \frac{d\rho}{dz} \frac{dp}{dz} + \omega^2 \rho^2 p = 0$$

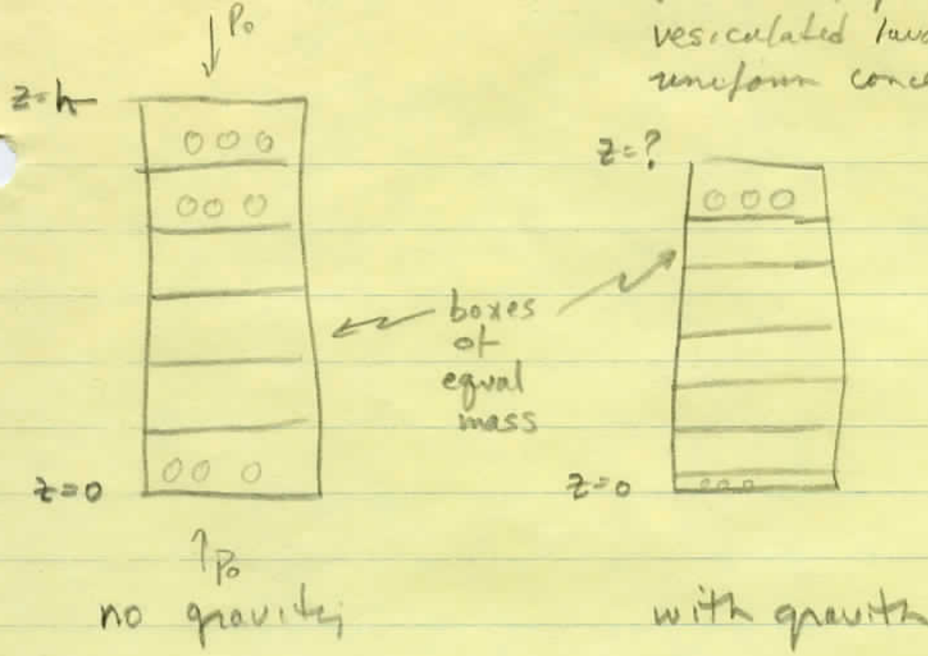
$$\frac{d^2 p}{dz^2} - \frac{1}{\rho} \frac{d\rho}{dz} \frac{dp}{dz} + \omega^2 \frac{\rho}{\lambda} p = 0$$

$$\frac{d^2 p}{dz^2} + f_1(z) \frac{dp}{dz} + \omega^2 f_2(z) p = 0$$

$$f_1(z) = -\frac{1}{\rho} \frac{d\rho}{dz} \qquad \int \frac{d\rho}{\rho} = \int f_1(z)\, dz = \ln(\rho)$$

$$f_2(z) = \frac{\rho}{\lambda} \qquad \rho = \rho_0 e^{\int_0^z f_1(z')\, dz'} \qquad \lambda = \frac{\rho}{f_2(z)}$$

calculation of density profile in vesiculated lava tube having a uniform concentration of gas (ideal)



let boxes have unknown height dz and known mass dm

ideal gas law for vessicles $PV =$ constant

let boxes have volume $dV = dV_g + dV_f$

volume of vessicles at pressure P $\quad dV_g = \dfrac{P_0\, dV_g^0}{P}$

volume of incompressible fluid at pressure p $\quad dV_f = dV_f^0$

initial volumes V^0 given by porosity $\quad dV = \left((1-n_0) + \dfrac{n_0 P_0}{P} \right) dV^0$

height of box dz is just its volume (given unit area tube)

$$dz = \left\{ (1-n_0) + \frac{n_0 P_0}{P} \right\} dV^0$$

initial volume related to initial bulk density and mass $dV^0 = \dfrac{dm}{\rho_b^0}$

$$dz = \left\{ (1-n_0) + \frac{n_0 P_0}{P} \right\} \frac{dm}{\rho_b^0}$$

pressure at any point weight of overlying material $\quad P = P_0 + g\rho_b^0 h - gm$

where $m = \int_0^m dm \equiv$ mass of material below box

$$dz = \frac{1}{\rho_b^0} \left\{ (1-n_0) + \frac{n_0 P_0}{P_0 + \rho_b^0 g h - gm} \right\} dm$$

$$\int_0^z dz \approx z = \int_0^m \frac{1}{\rho_b^0}\left\{(1-n_0) + \frac{n_0 p_0}{p_0 + \rho_b^0 gh - gm}\right\} dm$$

This gives

int. const.

$$z = \frac{(1-n_0)}{\rho_b^0} m - \frac{n_0 p_0}{\rho_b^0 g} \log\left\{(p_0 + \rho_b^0 gh) - gm\right\} + \frac{n_0 p_0}{\rho_b^0 g}\log(p_0 + \rho_b^0 gh)$$

new bulk density $= \left(\frac{dm}{dz}\right)$

$$\boxed{\rho_{bulk}(z) = \frac{dm}{dz} = \rho_b^0\left\{(1-n_0) + \frac{n_0 p_0}{(p_0 + \rho_g^0 gh) - gm(z)}\right\}^{-1}}$$

where $m(z)$ is such that

$$z(m) = \frac{(1-n_0)}{\rho_b^0} m + \frac{n_0 p_0}{\rho_b^0 g}\log\left\{\frac{(p_0 + \rho_g^0 gh) - gm}{(p_0 + \rho_g^0 gh)}\right\}$$

$z =$ top $m = \rho_b^0 h$

$$z_{top} = (1-n_0)h - \frac{n_0 p_0}{\rho_b^0 g}\left\{\log(p_0) - \log(p_0 + \rho_b^0 gh)\right\}$$

$$z_{top} = (1-n_0)h + \frac{n_0 p_0}{\rho_b^0 g}\log\left\{1 + \frac{\rho_g^0 gh}{p_0}\right\}$$

$\lim g \to 0 \quad z = h$

$\lim n_0 \to 0 \quad z = h$

gas	n	ρ_g	λ_g
porosity fluid	$(1-n)$	ρ_f	λ_f

$$\frac{\text{mass}}{\text{volume}} = \frac{n\rho_g + (1-n)\rho_f}{1}$$

$$\rho_{bulk} = n\rho_g + (1-n)\rho_f$$

$$= \rho_f - (\rho_f - \rho_g)n$$

$$= \rho_f - \Delta\rho\, n$$

$$p = -\lambda\, \nabla \cdot u$$

$$\nabla \cdot u = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \text{volumetric strain}$$

$$\text{volumetric strain} = \frac{\text{new volume} - \text{orig volume}}{\text{orig volume}}$$

gas	n	λ_g
fluid	$(1-n)$	λ_f

orig volume = 1

suppose pressure is p. Then since pressure uniform then volume strain in gas is $\nabla \cdot u = -p\lambda_g^{-1}$ and strain in fluid is $\nabla \cdot u = -p\lambda_f^{-1}$. so

$$-p\lambda_g^{-1} = \frac{-n + V^g_{new}}{n} \qquad -p\lambda_f^{-1} = \frac{-(1-n) + V^f_{new}}{(1-n)}$$

$$V^g_{new} = -np\lambda_g^{-1} + n \qquad V^f_{new} = -(1-n)p\lambda_f^{-1} + (1-n)$$

$$\text{total new volume} = -np\lambda_g^{-1} - (1-n)p\lambda_f^{-1} + 1$$

$$\text{bulk strain} = -p\left[\frac{n}{\lambda_g} + \frac{(1-n)}{\lambda_f}\right] \qquad \lambda_{bulk} = \left[\frac{n}{\lambda_g} + \frac{(1-n)}{\lambda_f}\right]^{-1}$$

$$\rho_{bulk} = \rho_f - [\rho_f - \rho_g]n \qquad \lambda_{bulk} = \left[\frac{1}{\lambda_f} + \left(\frac{1}{\lambda_g} - \frac{1}{\lambda_f}\right)n\right]^{-1}$$

Suppose
$$\rho_{bulk} = a + b\,n$$
$$\lambda^{-1}_{bulk} = c + d\,n$$

$$f_1 = -b\,(a+bn)^{-1}\,\frac{dn}{dz}$$

$$f_2 = (a+bn)(c+dn) = ac + (ad+bc)\,n + bd\,n^2$$

× note: Hildebrand, Meth of App Math p 145

given $\frac{d}{dx}\left(p\,\frac{dy}{dx}\right) + qy + \lambda r y = 0$ y prescribed at ends or $y' = 0$ at ends

Then $\lambda = \dfrac{\int_a^b (p y'^2 - q y^2)\,dx}{\int_a^b r y^2\,dx}$ and λ stationary w.r.t. variations in y.

in acoustics $\nabla \cdot u = -\frac{p}{\lambda}$ $\ddot{u} = -\frac{1}{\rho}\nabla p$

$$\nabla \cdot \left(\frac{1}{\rho}\nabla p\right) + \frac{\omega^2}{\lambda}\,p = 0$$

$$\frac{d}{dz}\left(\frac{1}{\rho}\,\frac{dp}{dz}\right) + \frac{1}{\lambda}\,\omega^2 p = 0$$

$$\omega^2 = \frac{\int \frac{1}{\rho}\left(\frac{dp}{dz}\right)^2 dz}{\int \frac{1}{\lambda}\,p^2\,dz} = \frac{I_1}{I_2}$$

$$\omega^2 = I_1 \, I_2^{-1} \qquad \text{let } p = \sum c_i \phi_i$$

$$\frac{\partial \omega^2}{\partial c_i} = 0 = \frac{\frac{\partial I_1}{\partial c_i} I_2 - I_1 \frac{\partial I_2}{\partial c_i}}{I_2^2}$$

assume $I_2 \neq 0$

$$0 = \frac{\partial I_1}{\partial c_i} I_2 - I_1 \frac{\partial I_2}{\partial c_i}$$

divide by I_2

$$0 = \frac{\partial I_1}{\partial c_i} - \omega^2 \frac{\partial I_2}{\partial c_i}$$

let $I_{2\,ij} = \int \frac{1}{\lambda} \phi_i \phi_j \, dz \qquad I_{1\,ij} = \int \frac{1}{\rho} \phi_i' \phi_j' \, dz$

$$I_1 = \sum_j \sum_k c_j c_k I_{1jk} \qquad I_2 = \sum_j \sum_k c_j c_k I_{2jk}$$

$$\frac{\partial I_1}{\partial c_i} = 2 \sum_k c_k I_{1ik} \qquad \frac{\partial I_2}{\partial c_i} = 2 \sum_k c_k I_{2ik}$$

$$\begin{bmatrix} I_{111} & I_{112} & I_{113} & \cdots \\ I_{121} & & & \\ & & & \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix} = \omega^2 \begin{bmatrix} I_{211} & \cdots \\ \vdots & \\ \vdots & \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix}$$

$$\underline{\underline{I}}_1 \, \underline{c} = \omega^2 \, \underline{\underline{I}}_2 \, \underline{c}$$

$$\boxed{\underline{\underline{I}}_2^{-1} \, \underline{\underline{I}}_1 \, \underline{c} = \omega^2 \, \underline{c}}$$

$$p = \sum_{n=1}^{N} c_n \cos\left[(n-\tfrac{1}{2}) \pi z / h \right]$$

$$\phi_n = \cos\left[\frac{(n-1/2)\pi}{h} z \right]$$

$$\phi_n' = -\frac{(n-1/2)\pi}{h} \sin\left[\frac{(n-1/2)\pi}{h} z \right]$$

$$I_{2ij} = \int_0^h \frac{1}{\lambda} \cos\left[\frac{(i-1/2)\pi z}{h} \right] \cos\left[\frac{(j-1/2)\pi z}{h} \right] dz$$

$$I_{1ij} = \frac{(i-1/2)(j-1/2)\pi^2}{h^2} \int_0^h \frac{1}{\rho} \sin\left[\frac{(i-1/2)\pi z}{h} \right] \sin\left[\frac{(j-1/2)\pi z}{h} \right] dz$$

note $\underline{\underline{I}}_1$, $\underline{\underline{I}}_2$ symmetric.

for homogeneous media

$$I_{2ii} = \frac{1}{\lambda}\int_0^h \cos^2\left[\frac{(i-1/2)\pi z}{h}\right] dz = \frac{1}{\lambda}\frac{h}{2}$$

$$I_{1ii} = \frac{\pi^2(i-1/2)^2}{h^2}\frac{1}{\rho}\int_0^h \sin^2\left[\frac{(i-1/2)\pi z}{h}\right] dz = \frac{\pi^2(i-1/2)^2}{h^2}\frac{1}{\rho}\frac{h}{2}$$

$$\frac{\pi^2}{2}\left(\frac{1}{2}\right)^2 = 1.234$$
$$\frac{\pi^2}{2}\left(\frac{3}{2}\right)^2 = 11.109$$
$$\frac{\pi^2}{2}\left(\frac{5}{2}\right)^2 = 30.842$$

$$\underline{\underline{I}}_1 \underline{c} = \omega^2 \underline{\underline{I}}_2 c$$

matrices diagonal, therefore:

$$\omega_i^2 = \frac{I_{1ii}}{I_{2ii}} = \frac{\lambda}{\rho}\frac{\pi^2(i-1/2)^2}{h^2}$$

$$\omega_i = \sqrt{\frac{\lambda}{\rho}}\frac{\pi(i-1/2)}{h} \qquad \alpha = \sqrt{\frac{\lambda}{\rho}}$$

$$= \frac{\alpha\pi}{h}(i-1/2)$$

from physical reasoning $\quad \alpha = \lambda_i f_i = \lambda_i \frac{\omega_i}{2\pi} \qquad \lambda_i = \frac{2h}{(i-1/2)}$

$$\omega_i = \frac{2\pi\alpha}{\lambda_i} = \frac{2\pi\alpha}{2}\frac{(i-1/2)}{h} = \frac{\alpha\pi}{h}(i-1/2)$$

which checks.

$$\omega_i = \frac{\alpha\pi}{h}(i-1/2)$$

$$f_i = \frac{\omega}{2\pi} = \frac{\alpha}{2h}(i-1/2)$$

if $\alpha = h = 1$ $\qquad f_1 = \frac{1}{4}$

$$f_2 = \frac{3}{2}\cdot\frac{1}{2} = \frac{3}{4}$$
$$f_3 = \frac{5}{2}\cdot\frac{1}{2} = \frac{5}{4}$$

set up problem in terms of perturbations.

let $\frac{1}{\rho} = \frac{1}{\rho_0} + f_\rho(z)$

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + f_\lambda(z)$$

$$I_{2ij} = \int_0^h \frac{1}{\lambda_0} \cos\left[\frac{(i-\frac{1}{2})\pi z}{h}\right]\cos\left[\frac{(j-\frac{1}{2})\pi z}{h}\right] dz + \int_0^h f_\rho(z) \cos\left[\frac{(i-\frac{1}{2})\pi z}{h}\right]\cos\left[\frac{(j-\frac{1}{2})\pi z}{h}\right] dz$$

$$= I^0_{2ij} + I^1_{2ij}$$

$$I_{1ij} = I^0_{1ij} + I^1_{2ij} \qquad ?$$

Test Case VARIATIONAL procedure

+

$h = 1.0$ km

$\times \frac{10^{15} g}{km^2 sec}$ $\times \frac{10^{15} g}{km^3}$

λ, ρ gas = 0.5, 0.5 ie $\alpha_{fluid} = 1.0$ km/sec

λ, ρ fluid = 18, 2 ie $\lambda_{gas} = 3.0$ km/sec

100 integration pts

	case 1		case 2		case 3	
	$p = 0.5$		$p = 0.4 + 0.2z$		$p = 0.2 + 0.6z$	
0	0.22056	].4414	0.23503	].43457	0.27186	].44281
1	0.66196		0.66960		0.71467	
2	1.1028		1.1108		1.1680	
3	1.5440		1.5530		1.6237	
4	1.9851	].4411	1.99555	].4427	2.0801	].4608
5	2.4262	].4411	2.4382		2.5409	
6	2.8673		2.8805		2.9961	

The 3d lava tube in all its glory

acoustic eqn $\nabla \cdot \left(\frac{1}{\rho} \nabla p \right) + \frac{\omega^2}{\lambda} p = 0$

$$\nabla p = \frac{\partial p}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial p}{\partial \phi}\hat{\phi} + \frac{\partial p}{\partial z}\hat{z}$$

$\rho = \rho(z)$
$\lambda = \lambda(z)$

$$\nabla \cdot \underline{f} = \frac{1}{r}\frac{\partial}{\partial r}(r f_r) + \frac{1}{r}\frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z}$$

$$\frac{1}{\rho}\nabla p = \frac{1}{\rho}\frac{\partial p}{\partial r}\hat{r} + \frac{1}{\rho r}\frac{\partial p}{\partial \phi}\hat{\phi} + \frac{1}{\rho}\frac{\partial p}{\partial z}\hat{z}$$

$$\nabla \cdot \left(\frac{1}{\rho}\nabla p\right) = \frac{1}{\rho r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{\rho r^2}\frac{\partial^2 p}{\partial \phi^2} + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial z}\right)$$

so equation is

$$\frac{1}{\rho r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{\rho r^2}\frac{\partial^2 p}{\partial \phi^2} + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial z}\right) + \frac{\omega^2}{\lambda} p = 0$$

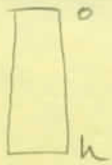
let $p = R(r)\,\Phi(\phi)\,Z(z)$

$$0 = \frac{1}{\rho r}\Phi(\phi)Z(z)\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\rho r^2}RZ\frac{\partial^2\Phi}{\partial\phi^2} + R\Phi\frac{d}{dz}\left(\frac{1}{\rho}\frac{dZ}{dz}\right) + \frac{\omega^2}{\lambda}R\Phi Z$$

divide through by p multiply by ρ rearrange terms

$$-\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{1}{r^2\Phi}\frac{d^2\Phi}{d\phi^2} = \frac{\rho}{Z}\frac{d}{dz}\left(\frac{1}{\rho}\frac{dZ}{dz}\right) + \frac{\rho\omega^2}{\lambda}$$

so these terms equal to a constant, say γ^2



input h = length of pipe

nh = number of sampling points from 0 to h

por = porosity, a function of h

$$\frac{\rho}{Z}\frac{d}{dz}\left(\frac{1}{\rho}\frac{dZ}{dz}\right) + \frac{\rho\omega^2}{\lambda} - \gamma^2 = 0$$

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\Phi}\frac{d^2\Phi}{\partial\phi^2} + \gamma^2 = 0$$

multiply by r^2 and rearrange

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{\partial R}{\partial r}\right) + r^2\gamma^2 = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

so These equal to a constant say m^2

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{\partial R}{\partial r}\right) + r^2\gamma^2 = m^2$$

$$\frac{d^2\Phi}{\partial\phi^2} = -m^2\Phi$$

this second equation can be solved to give $\Phi = e^{\pm im\phi}$
but since physically $e^{\pm im\phi} = e^{\pm i\lambda(\phi+2\pi)}$ we
must have $e^{\pm 2\pi mi} = 1$ or $m = 0, 1, 2, \ldots$

the r equation is Then $r\frac{dR}{dr} + r^2\frac{d^2R}{dr^2} + (r^2\gamma^2 - m^2)R = 0$
or:

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (r^2\gamma^2 - m^2)R = 0$$

comparing to the form Hildebrand gives:
$a=1$ $b=0$ $c=-m^2$ $d=\gamma^2$ $s=1$ and the soln is

$R = J_m(\gamma r)$ but the boundary conditions at the rigid surface of the cylinder are that the normal derivative of the pressure is zero. ie $\nabla p \cdot \hat{r} = \frac{\partial p}{\partial r} = 0$ so that we must choose γ so that this condition holds at $r = a$ = cylinder's surface. Suppose Π_{ml} give the value of ~~of~~ r of the l-th (maxima - minima) of the m-th bessel function $J_m(r)$. Then $\gamma a = \Pi_{ml}$ and

$$R = J_m\left(\frac{\Pi_{ml}}{a} r\right) \qquad \gamma = \frac{\Pi_{ml}}{a}$$

note. since $J_0(0r)$ satisfies eqn $\Pi_{00} = 0$

getting back to the Z equation. we have

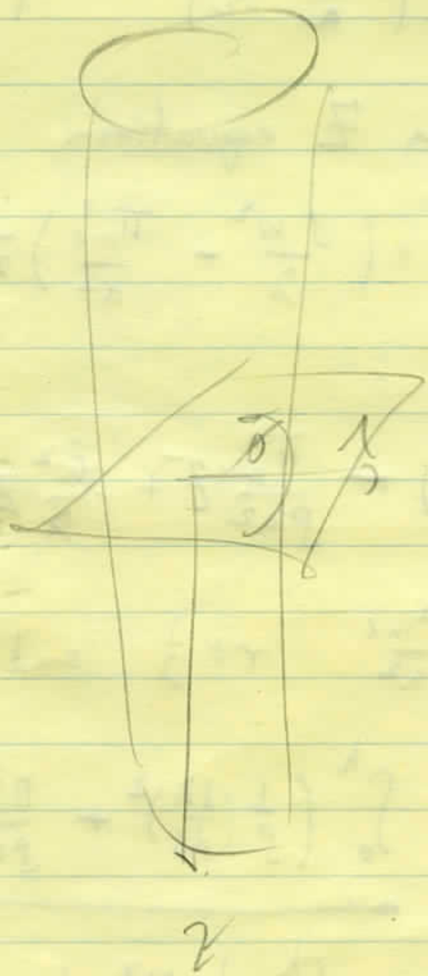
$$\rho \frac{d}{dz}\left(\frac{1}{\rho}\frac{dZ}{dz}\right) + \left(\frac{\rho\omega^2}{\lambda} - \frac{\Pi_{ml}^2}{a^2}\right) Z = 0$$

or

$$\frac{d}{dz}\left(\frac{1}{\rho}\frac{dZ}{dz}\right) - \frac{\Pi_{ml}^2}{\rho a^2} Z + \frac{\omega^2}{\lambda} Z = 0$$

note $p = \frac{1}{\rho}$ $q = -\frac{\Pi^2_{ml}}{\rho a^2}$ $r = \frac{1}{\lambda}$ and variational integral is

$$\omega^2 = \frac{\int_0^h \left(\frac{1}{\rho}\left(\frac{dZ}{dz}\right)^2 + \frac{\Pi_{ml}^2}{\rho a^2} Z^2\right) dz}{\int_0^h \frac{1}{\lambda} Z^2 \, dz}$$


θ
z

now suppose ρ, λ constants, then

$$\frac{d^2Z}{dz^2} + \left(\frac{\rho}{\lambda}\omega^2 - \frac{\Pi_{m\ell}^2}{a^2}\right) Z = 0$$

let $\delta^2 = \frac{\rho}{\lambda}\omega^2 - \frac{\pi^2_{m\ell}}{a^2}$ so that equation becomes

$$\frac{d^2Z}{dz^2} + \delta^2 Z = 0$$

solution with boundary conditions rigid at $z=0$, pressure free at $z=h$ is

$$Z = \cos(\delta z) = \cos\left(\frac{(n+\frac{1}{2})\pi}{h} z\right) \qquad n = 0, 1, 2 \dots$$

$$\delta^2 = \frac{(n+\frac{1}{2})^2\pi^2}{h^2} = \frac{\rho}{\lambda}\omega^2 - \frac{\Pi_{m\ell}^2}{a^2}$$

or

$$\omega = \alpha \sqrt{\frac{(n+\frac{1}{2})^2\pi^2}{h^2} + \frac{\Pi_{m\ell}^2}{a^2}}$$

$$\boxed{P_{n\ell m} = \cos\left[\frac{(n+\frac{1}{2})\pi z}{h}\right] J_m\left(\frac{\Pi_{m\ell}\, r}{h}\right) e^{\pm i m \phi}}$$

$$\boxed{\omega = \alpha \sqrt{\frac{(n+\frac{1}{2})^2\pi^2}{h^2} + \frac{\Pi_{m\ell}^2}{a^2}}}$$

$n, m, \ell = 0, 1, 2 \dots$

in order of size

mn	Time
0,0	0.0000
1,0	1.8413
2,0	3.0544
0,1	3.8316
3,0	4.2012
4,0	5.3172
1,1	5.3315
4,0	6.4122
2,1	6.7067
0,3	7.0015
5,0	7.4122
3,1	8.0178
7,0	8.4707
1,2	8.5363
4,1	9.2893
8,0	9.3325
2,2	9.9695

Table of $\Pi_{m\ell} \lesssim 10.0$

$m \backslash \ell$	0	1	2	3
0	0	3.8316	7.0155	10.1735
1	1.8412	5.3315	8.5363	11.7060
2	3.0544	6.7067	9.9695	13.1704
3	4.2012	8.0178	11.3462	14.5859
4	5.3172	9.2893	12.6828	15.9643
5	6.4122	10.5339	13.9894	17.3136
6	7.4777	11.7593	15.2727	
7	8.4707	14.1705		
8	9.3325	15.3621		
9	10.0616			
10				

setting up the eigenvalue problem

we have $0 = \frac{\partial I_1}{\partial c_i} - \omega^2 \frac{\partial I_2}{\partial c_i}$

$$I_1 = \int \frac{1}{\rho}\left(\frac{dZ}{dz}\right)^2 dz + \frac{\pi^2_{me}}{a^2}\int \frac{1}{\rho} Z^2 dz = I_{1A} + \frac{\pi^2_{me}}{a^2} I_{1B}$$

$$I_2 = \int \frac{1}{\lambda} Z^2 dz$$

let $I_{1A_{ij}} = \int_0^h \frac{1}{\rho} \phi_i' \phi_j' dz \qquad I_{1B_{ij}} = \int_0^h \frac{1}{\rho} \phi_i \phi_j dz$

$$I_{2ij} = \int_0^h \frac{1}{\lambda} \phi_i \phi_j dz$$

$$\frac{\partial I_1}{\partial c_i} = \frac{\partial I_{1A}}{\partial c_i} + \frac{\pi^2_{me}}{a^2}\frac{\partial I_{1B}}{\partial c_i}$$

$I_{1A} = \sum_j \sum_k c_j c_k I_{1Ajk}$ so $\frac{\partial I_{1A}}{\partial c_i} = 2\sum_k c_k I_{1Aik}$

and so forth for I_{1B} and I_2

$$\frac{\partial I_{1B}}{\partial c_i} = 2\sum_k c_k I_{1Bik} \qquad \frac{\partial I_2}{\partial c_i} = 2\sum_k c_k I_{2ik}$$

$$\begin{bmatrix} I_{1A_{11}} & I_{1A_{12}} & \cdots \\ \vdots & & \\ & & \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \frac{\pi^2_{me}}{a^2}\begin{bmatrix} I_{1B_{11}} & \cdots \\ \vdots & \\ & \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ \\ \end{bmatrix} = \omega^2\begin{bmatrix} I_{211} & \cdots \\ \vdots & \\ & \end{bmatrix}\begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$\left(\underline{\underline{I}}_{1A} + \frac{\pi^2_{me}}{a^2}\underline{\underline{I}}_{1B}\right)\underline{c} = \omega^2 \underline{\underline{I}}_2 \underline{c}$$

$$I_{1A_{ij}} = \frac{(i-\frac{1}{2})(j-\frac{1}{2})\pi^2}{h^2} \int_0^h \frac{1}{\rho} \sin\left[\frac{(i-\frac{1}{2})\pi z}{h}\right] \sin\left[\frac{(j-\frac{1}{2})\pi z}{h}\right] dz$$

$$I_{1B_{ij}} = \int_0^h \frac{1}{\rho} \cos\left[\frac{(i-\frac{1}{2})\pi z}{h}\right] \cos\left[\frac{(j-\frac{1}{2})\pi z}{h}\right] dz$$

$$I_{2_{ij}} = \int_0^h \frac{1}{\lambda} \cos\left[\frac{(i-\frac{1}{2})\pi z}{h}\right] \cos\left[\frac{(j-\frac{1}{2})\pi z}{h}\right] dz$$

$$\underline{\underline{I}}_2^{-1} \left(\underline{\underline{I}}_{1A} + \frac{\pi_{me}^2}{a^2} \underline{\underline{I}}_{1B} \right) \underline{c} = \omega^2 \underline{c}$$

$$\left[\left(\underline{\underline{I}}_2^{-1} \underline{\underline{I}}_{1A} \right) + \frac{\pi_{me}^2}{a^2} \left(\underline{\underline{I}}_2^{-1} \underline{\underline{I}}_{1B} \right) \right] \underline{c} = \omega^2 \underline{c}$$

frequency for 1.0 × 1.4 $\alpha=1,\ h=1,\ a=\frac{1}{5}$ $f=\frac{1}{2\pi}\sqrt{(n+\frac{1}{2})^2\pi^2+25\,\Pi_{mc}^2}$