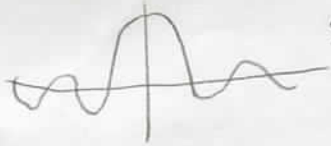

Sinc

MRN030 9/3/93

What is the envelope and instantaneous frequency of $x(t) = \frac{\sin \omega t}{t}$?

$$\mathcal{H}x = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \omega t'}{t'} \frac{1}{t-t'}\, dt' \qquad \text{but} \int_{-\infty}^{\infty} \frac{\sin ax}{x(x-b)}\, dx = \pi \frac{\cos ab - 1}{b}$$

see G&R 3.723.12.

$$\text{so} \quad \mathcal{H}x = \frac{\cos \omega t - 1}{t} = x^*$$

Then the envelope $E = \sqrt{x^2 + (\mathcal{H}x)^2}$:

$$E^2 = \frac{1}{t^2}\left[\sin^2 \omega t + \cos^2 \omega t + 1 - 2\cos \omega t \right]$$

$$= \frac{2}{t^2}\left[1 - \cos \omega t \right]$$

note $\lim_{t \to 0} E = \frac{2}{t^2}\left(\frac{t^2}{2} + O(t^4) \right) = 1$ from taylor exp of cos.

and the instantaneous frequency

$$\omega(t) = \frac{x\dot{x}^* - \dot{x}x^*}{E^2} \quad \text{but} \quad \begin{aligned} \dot{x} &= \frac{d}{dt} t^{-1}\sin \omega t = \omega t^{-1}\cos \omega t - t^{-2}\sin \omega t \\ \dot{x}^* &= \frac{d}{dt} t^{-1}(\cos \omega t - 1) = -\omega t^{-1}\sin \omega t - (\cos \omega t - 1)\, t^{-2} \end{aligned}$$

$$\omega(t) = E^{-2}\left\{ -\omega t^{-2}\sin^2 \omega t \cancel{- t^{-3}\sin \omega t(\cos \omega t - 1)} - \omega t^{-2}\cos \omega t(\cos \omega t - 1) \cancel{+ t^{-3}\sin \omega t(\cos \omega t - 1)} \right\}$$

$$= E^{-2}\left\{ -\omega t^{-2}(\sin^2 \omega t + \cos^2 \omega t) + \omega t^{-2}\cos \omega t \right\}$$

$$= E^{-2}\omega \underbrace{t^{-2}\left[\cos \omega t - 1 \right]}_{\frac{E}{2}} = -\frac{\omega}{2}$$

check this numerically w DXSH.
seems ok.