Ideal gas law

Planetary Atmospheric density

MRN050

$$pV = nRT \Rightarrow \frac{P}{RT} = \frac{n}{V}$$

$n = \#$ moles, let $m =$ mass/mole then

$$\rho = \frac{mn}{V} = \frac{m}{RT} P = \alpha p$$

where $\alpha = \frac{m}{RT}$ scales pressure to density

stress equilibrium $T_{ij,j} + f_i = 0$ $f_i =$ force per unit volume

$T_{ij} = -p\delta_{ij}$ in fluid so

$-p_{,i} + f_i = 0$ That is $\nabla p = \underline{f}$

gravitation in radially symmetric object

$\underline{f} = \rho g_r \hat{r}$ $g_r =$ radial acceleration

$$\frac{\partial p}{\partial r} = \rho g_r = \alpha p g_r$$

massive planet with tenuous atmosphere

$$g_r = -g_s \frac{r_s^2}{r^2}$$

$g_s =$ surface acceleration

$r_s =$ surface radius

$$\frac{dp}{dr} = -g_s r_s^2 r^{-2} \rho = -g_s r_s^2 \alpha r^{-2} p$$

$$\frac{dp}{p} = -c r^{-2} dr \quad \text{where } c = g_s r_s^2 \alpha$$

particular soln

$$\frac{dp}{p} = -c r^{-2} dr$$

$$\ln p = c r^{-1} + \ln a \qquad a = \text{const}$$

$$p = a e^{c/r}$$

general soln note constant solves eqn so

$$p = a e^{c/r} + b e^{c/R}$$

boundary condition $p(r = r_s) = p_0$ $\quad p(r \to \infty) = 0$

$$0 = a e^{0} + b e^{c/R} \qquad \text{so} \quad a = -b e^{c/R}$$

$$p_0 = -b e^{c/r_s} e^{c/r} + b e^{c/r_s}$$

$$= b e^{c/r_s} \left(e^{c/r} - 1 \right) \qquad b = \frac{p_0}{(e^{c/r_s} - 1) e^{c/r_s}}$$

$$\boxed{p = \frac{p_0}{(e^{c/r_s} - 1)} \left(e^{c/r} - 1 \right)}$$

now suppose $r = r_s + z$ with $|z| \ll r_s$

$$r^{-1} = (r_s + z)^{-1} = r_s^{-1} \left(1 + \frac{z}{r_s}\right)^{-1} \approx r_s^{-1} \left(1 - \frac{z}{r_s}\right)$$

$$e^{c r^{-1}} = e^{c \left(r_s^{-1} - \frac{z}{r_s^2}\right)} = e^{c/r_s} e^{-cz/r_s^2}$$

$$p = \frac{p_0}{e^{c/r_s} - 1}\left(e^{c/r_s} e^{-cz/r_s^2} - 1\right)$$

$$= \frac{p_0}{e^{g_s \alpha r_s} - 1}\left(e^{g_s \alpha r_s} e^{-g_s \alpha z} - 1\right)$$

in lim $r_s \to \infty$

$$p = p_0 e^{-g_s \alpha z}$$

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now suppose self-gravitation of sphere of gas

$$g_r = -\frac{\gamma M(r)}{r^2} \quad \text{where } M(r) = 4\pi \int_0^r \rho(r') r'^2 dr'$$

so $$\frac{dp}{dr} = -\frac{\alpha \gamma p}{r^2}\int_0^r \rho(r') r'^2 dr'$$

$$\frac{1}{p}\frac{dp}{dr} = -\frac{\alpha^2 \gamma}{r^2}\int_0^r p(r') r'^2 dr'$$

$$r^2 \frac{d}{dr} \ln p = -\alpha^2 \gamma \int_0^r p(r') r'^2 dr'$$

$$\frac{d}{dr} r^2 \frac{d}{dr} \ln p = -\alpha^2 \gamma r^2 p$$