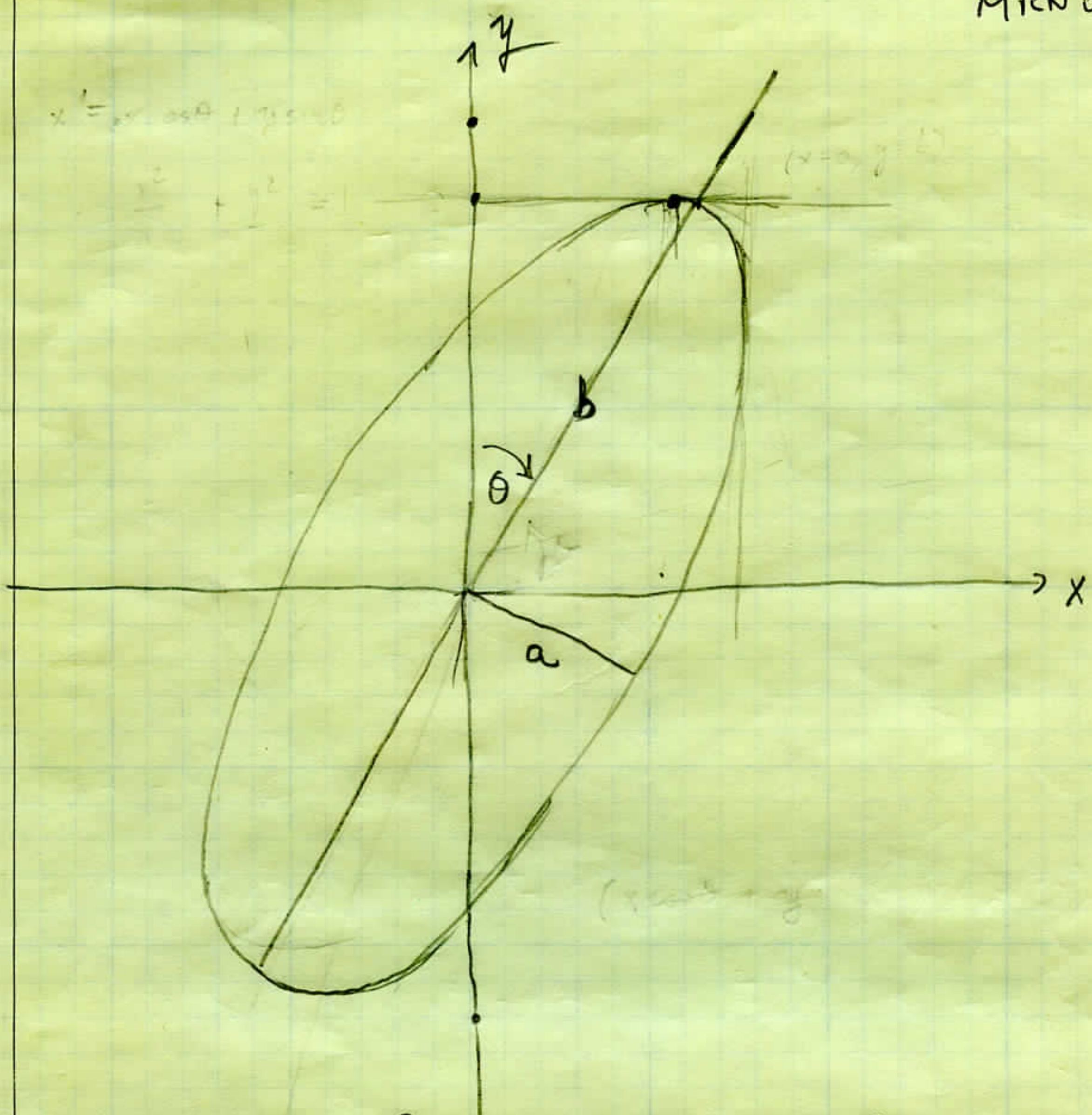
MRN053

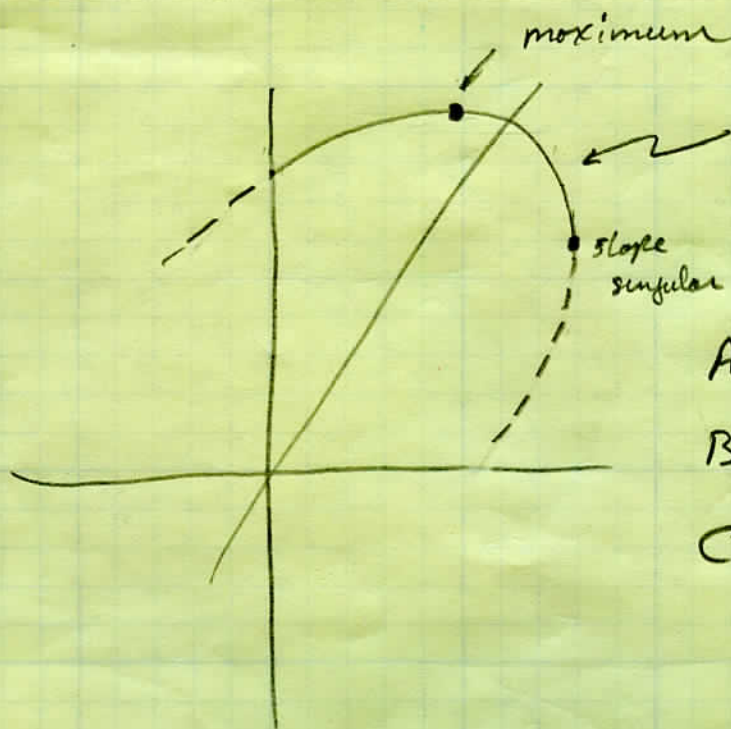


$$\frac{(x\cos\theta - y\sin\theta)^2}{a^2} + \frac{(y\cos\theta + x\sin\theta)^2}{b^2} - 1 = 0$$

$$\frac{x^2\cos^2\theta}{a^2} + \frac{y^2\sin^2\theta}{a^2} - \frac{2xy\cos\theta\sin\theta}{a^2} +$$

$$\frac{y^2\cos^2\theta}{b^2} + \frac{x^2\sin^2\theta}{b^2} + \frac{2xy\cos\theta\sin\theta}{b^2} - 1 = 0$$

$$y^2\underbrace{\left\{\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)\right\}}_{A} + \underbrace{\left\{2\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cos\theta\sin\theta\, x\right\}}_{B}y + \underbrace{\left\{x^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) - 1\right\}}_{C}$$



$$y = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

$A = O(x^0)$

$B = O(x^1)$

$C = O(x^2)$

$$\frac{\partial y}{\partial x} = \frac{-\frac{\partial B}{\partial x} + \left(2B\frac{\partial B}{\partial x} - 4A\frac{\partial C}{\partial x}\right)\left(B^2 - 4AC\right)^{-1/2}}{2A} = 0$$

$$\left(\frac{\partial B}{\partial x}\right)^2 \left(B^2 - 4AC\right) = \left(2B\frac{\partial B}{\partial x} - 4A\frac{\partial C}{\partial x}\right)^2$$

$$\underset{\uparrow}{} \quad$$

$$O(x^0) \quad O(x^2) = \left(O(x) \quad O(x)\right)^2$$

quadratic in x w/ no x' term

$$0 = -\cancel{B^2\left(\frac{\partial B}{\partial x}\right)^2} + 4AC\left(\frac{\partial B}{\partial x}\right)^2 + \overset{3}{\cancel{4}}B^2\left(\frac{\partial B}{\partial x}\right)^2 + 16A^2\left(\frac{\partial C}{\partial x}\right)^2 - 16AB\frac{\partial B}{\partial x}\frac{\partial C}{\partial x}$$

$$4AC\left(\frac{\partial B}{\partial x}\right)^2 = 4\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)4\left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2 \cos^2\theta \sin^2\theta \cdot \left\{x^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) - 1\right\}$$

$$3B^2\left(\frac{\partial B}{\partial x}\right)^2 = 3 \cdot \left(\frac{1}{b^2} - \frac{1}{a^2}\right)^4 \cos^4\theta \sin^4\theta \quad x^2$$

$$16A^2\left(\frac{\partial C}{\partial x}\right)^2 = 16 \cdot \left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)^2 \cdot 4 \cdot \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)^2 x^2$$

$$-16\,A B \frac{\partial B}{\partial x}\frac{\partial C}{\partial x} = -16\cdot\left(\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2}\right)\cdot 4\cdot\left(\frac{1}{b^2}-\frac{1}{a^2}\right)^2 \cos^2\theta \sin^2\theta \cdot$$

$$2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right) x^2$$

yech. let

$$A = A_0$$

$$B = B_1 x \qquad \frac{\partial B}{\partial x} = B_1$$

$$C = C_0 + C_2 x^2 \qquad \frac{\partial C}{\partial x} = 2 C_2 x$$

$$4AC\left(\frac{\partial B}{\partial x}\right)^2 = 4 A_0 \left(C_0 + C_2 x^2\right) B_1^2$$

$$= 4 A_0 B_1^2 C_0 + 4 A_0 B_1^2 C_2 x^2$$

$$3 B^2 \left(\frac{\partial B}{\partial x}\right)^2 = 3 B_1^4 x^2$$

$$16 A^2 \left(\frac{\partial C}{\partial x}\right)^2 = 16 A_0^2 4 C_2^2 x^2 = 64 A_0^2 C_2^2 x^2$$

$$-16 A B \frac{\partial B}{\partial x}\frac{\partial C}{\partial x} = -16 A_0 B_1 x B_1 2 C_2 x = -32 A_0 B_1^2 C_2$$

$$\left(\cancel{4 A_0 B_1^2 C_2} + 3 B_1^4 + 64 A_0^2 C_2^2 - \overset{28}{\cancel{32}} A_0 B_1^2 C_2\right) x^2 =$$

$$- 4 A_0 B_1^2 C_0$$

$=1$

$$x^2 = \frac{4 A_0 B_1^2}{3 B_1^4 + 64 A_0^2 C_2^2 - 28 A_0 B_1^2 C_2}$$

$\frac{\partial y}{\partial x} = \infty$ when $B^2 - 4AC = 0$

solve for x

$$B_1^2 x^2 - 4A_0(C_0 + C_2 x^2) = 0$$

$$B_1^2 x^2 - 4A_0C_0 - 4A_0C_2x^2 = 0$$

$$(B_1^2 - 4A_0C_2)x^2 = 4A_0C_0$$

$$x^2 = \frac{4A_0C_0}{(B_1^2 - 4A_0C_2)} =$$

$$= \frac{-4\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)}{4\left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2 \cos^2\theta \sin^2\theta - 4\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)}$$

checked by computer. seems to work

$$\boxed{x^2 = \frac{\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)}{\left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) - \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cos^2\theta \sin^2\theta}}$$

$B^2 - 4AC$

$\frac{dy}{dx} = \infty$ when $-\frac{1}{4}B^2 + AC = 0$ solve for $\cos^2\theta$ (quadratic eqn)

$\sin^2\theta = 1 - \cos^2\theta$

$$-\frac{1}{4}B^2 = -\frac{1}{4}4\left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2 \cos^2\theta\,(1-\cos^2\theta)\,x^2$$

$$= -\left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2 x^2\cos^2\theta + \left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2 x^2\cos^4\theta$$

$$A = \frac{1}{a^2} - \frac{\cos^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cos^2\theta + \frac{1}{a^2}$$

$$C = \left(\frac{\cos^2\theta}{a^2} + \frac{1}{b^2} - \frac{\cos^2\theta}{b^2}\right)x^2 - 1$$

$$= \left(\frac{1}{a^2} - \frac{1}{b^2}\right)x^2\cos^2\theta + \left(\frac{x^2}{b^2} - 1\right)$$

$$AC = \left\{\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cos^2\theta + \frac{1}{a^2}\right\}\left\{\left(\frac{1}{a^2} - \frac{1}{b^2}\right)x^2\cos^2\theta + \left(\frac{x^2}{b^2} - 1\right)\right\}$$

$$= -\left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2 x^2\cos^4\theta + \frac{1}{a^2}\left(\frac{x^2}{b^2} - 1\right)$$

$$+ \underbrace{\left\{\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\left(\frac{x^2}{b^2} - 1\right) + \frac{1}{a^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)x^2\right\}}_{\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\left\{\frac{x^2}{b^2} - \frac{x^2}{a^2} - 1\right\}}\cos^2\theta$$

$$\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\left\{\frac{x^2}{b^2} - \frac{x^2}{a^2} - 1\right\}\cos^2\theta$$

Terms multiplying

$\underline{\cos^4\theta}$:

$$\left(\frac{1}{b^2}-\frac{1}{a^2}\right)^2 x^2 - \left(\frac{1}{b^2}-\frac{1}{a^2}\right)^2 x^2$$

$$= 0$$

$\underline{\cos^2\theta}$: $$-\left(\frac{1}{b^2}-\frac{1}{a^2}\right)^2 x^2 + \left(\frac{1}{b^2}-\frac{1}{a^2}\right)\left\{\frac{x^2}{b^2}-\frac{x^2}{a^2}-1\right\}$$

$\underline{\cos^0\theta}$: $$\frac{1}{a^2}\left(\frac{x^2}{b^2}-1\right)$$

$$x^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\left\{-\cancel{\frac{1}{b^2}}+\cancel{\frac{1}{a^2}}+\cancel{\frac{1}{b^2}}-\cancel{\frac{1}{a^2}}-\frac{1}{x^2}\right\}$$

$$= -\left(\frac{1}{b^2}-\frac{1}{a^2}\right)$$

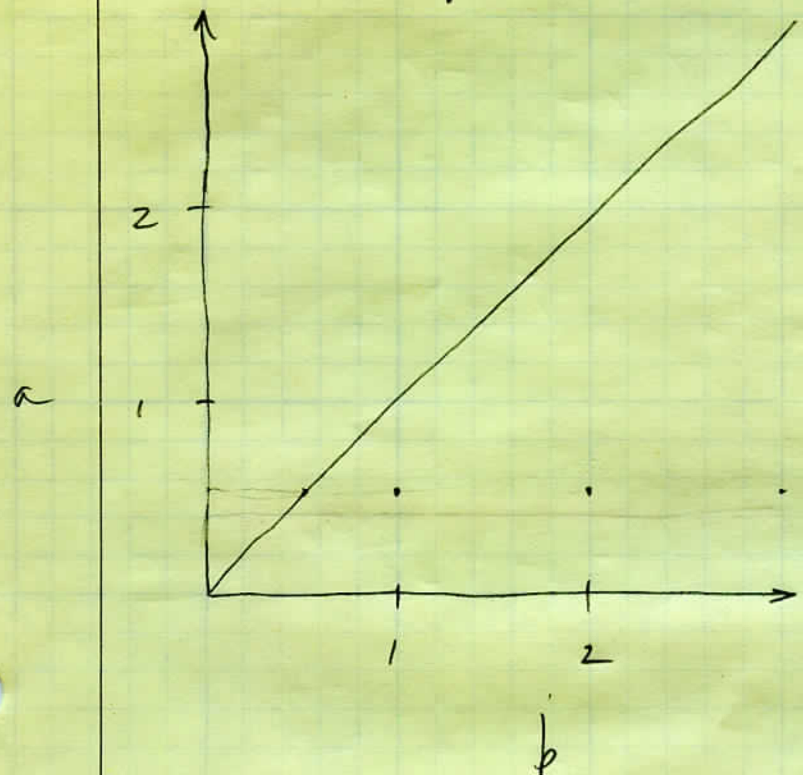
checked by computer, seems to work.

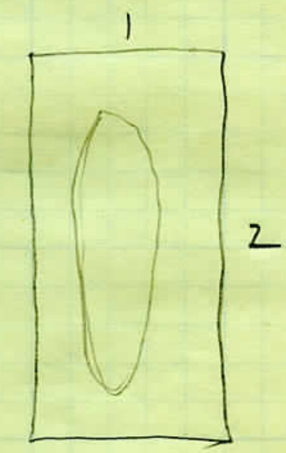
$$\boxed{\cos^2\theta = \frac{\frac{1}{a^2}\left(\frac{x^2}{b^2}-1\right)}{\left(\frac{1}{b^2}-\frac{1}{a^2}\right)}}$$

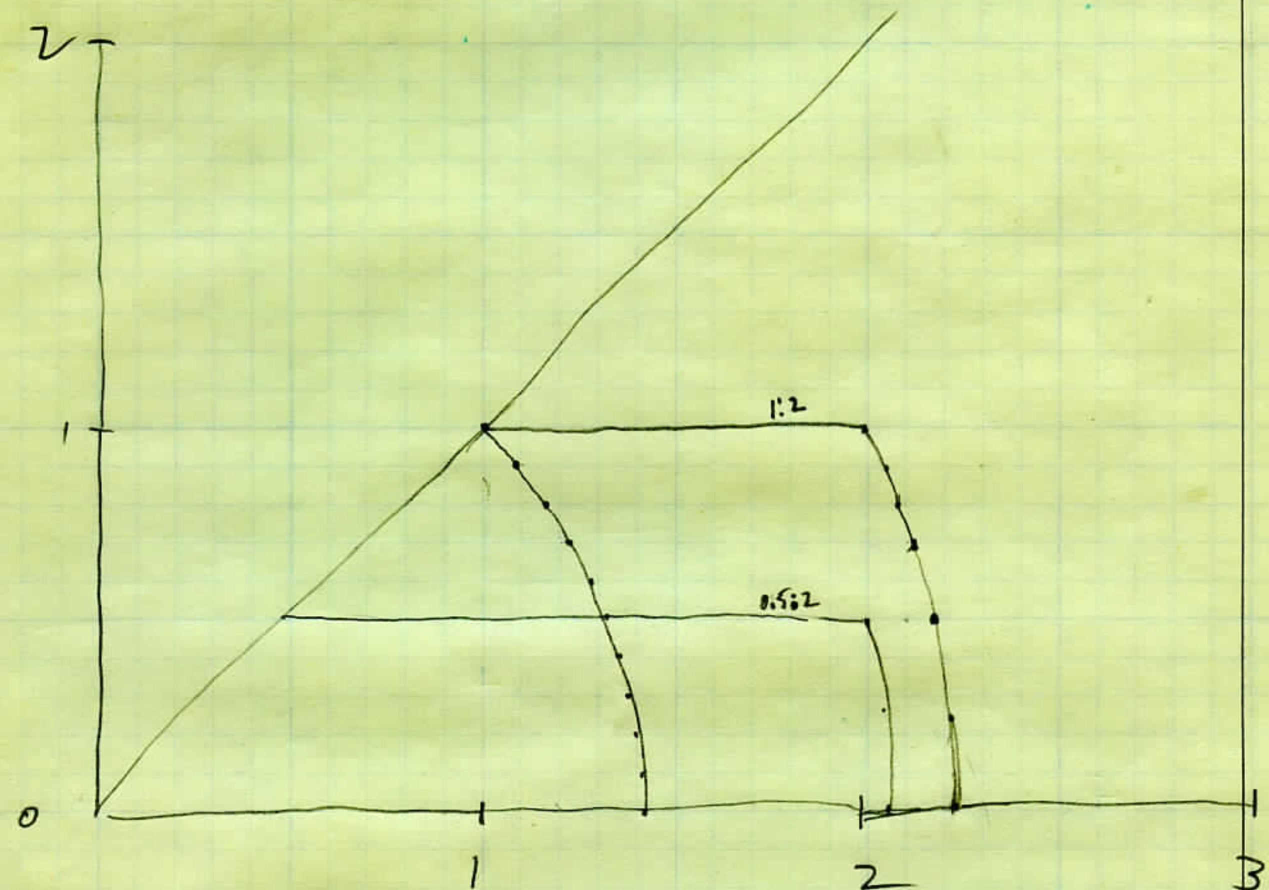
$\theta = 30^\circ \qquad X_{inf} = 1.3228$

$\theta = 10^\circ \qquad X_{inf} = 1.04425$

$\theta = 20^\circ \qquad X_{inf} = 1.16230$






2
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0
1
2
3
1:2
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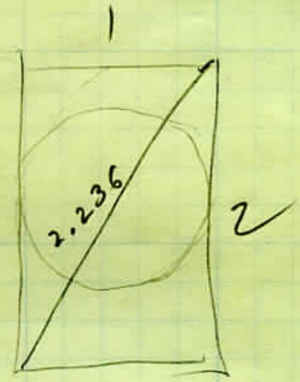
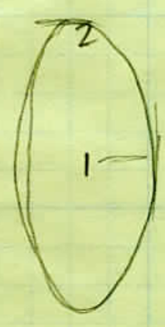

1
2.236
2


2
1