MRN068

Statistics of detects using cross-correlation
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signal A_i of length N. it has mean-squared amplitude a^2 with $\sum_i A_i^2 = Na^2$
(A_i has zeo mean)

signal B_i also of length N with mean-squared amplitude b^2 with $\sum_i B_i^2 = Nb^2$
B_i has zero mean

noise N_i of length N, uncorrelated, normally distributed, zero mean, variance n^2

consider cross-correlation

$$\sum_i^N A_i (B_i + N_i) = \overbrace{\sum_i^N A_i B_i}^{\text{Term 1}} + \overbrace{\sum_i^N A_i N_i}^{\text{Term 2}}$$

now assume that $A_i \propto B_i$ (same signals, different ampl)
and assume A_i is uncorrelated w.r.t. N_i, so
it can be treated as r.v. in second term.
Then first term has constant value Nab and
second term has variance $= Na^2 n^2$.

Let's say that a detection occurs when

$$\text{Term 1} > 2\sqrt{\text{Variance of term 2}}$$ ie 95% conf

or

$$Nab > 2\sqrt{N}\, an$$

or

$$\frac{1}{2}\sqrt{N}\,\frac{b}{n} > 1$$

Effect of Bandwidth

note correlation between $(A_i N_i)$ and (A_{i+1}, N_{i+1}) will make Term 2 larger than the uncorrelated case since $Var(Z_1 + Z_2) = Var(Z_1) + Var(Z_2) + 2cov(Z_1, Z_2)$.

Special (extreme) case of complete correlation. Note $var(cZ) = c^2 var(Z)$ so ~~the~~ if $Z_i = A_i B_i$ and all's Z's are fully correlated

$$Var(\Sigma Z_i) = Var(N Z_i) = N^2 Var(Z_i) = N^2 a^2 n^2$$

in this case, the discrimination condition is

$$Nab > 2Nan$$

or

$$\frac{1}{2} \cdot \frac{b}{n} > 1$$

That is, there is no improvement in discrimination with number of points

in the more general case of bandwith, ω, where $\omega = 0$ means fully correlated and $\omega = 1$ means fully uncorrelated, we might expect a formula something like

$$\frac{1}{2} N^{\frac{\omega}{2}} \frac{b}{n} > 1$$

which atleast has the right limits. The exact formula will depend upon the exact definition of "bandwidth", or more exactly, the autocorrelation function of z_i.

This follows from

$$Var(\Sigma z_i) = [1, 1, \cdots] \; cov(z_i) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \sum_i \sum_j [cov(z)]_{ij}$$

suppose $[cov(z_i)]_{ij} \propto \exp\{-c|i-j|\}$

Then $Var(\Sigma z_i) = \sum_i \sum_j \exp\{-c|i-j|\}$

This can be divided into the diagonal part of the sum (that is $i=j$), which is just N, plus 2 times the upper triangle part (where $j > i$).

note along the first ~~the~~ row, this is $\sum_{j=1}^{N} \exp(-jc)$

$$= \sum_{j=1}^{N} (e^{-c})^j = \sum_{j=1}^{N} E^j \quad \text{with } E = e^{-c}$$

now Note G&R p1 0.112.0

$$\sum_{k=1}^{n} a q^{k-1} = \frac{a(q^n - 1)}{q-1}$$

with $a=1$ and $q=e^{-c}$ This gives $\sum_{k=1}^{N}(e^{-c})^{k-1} = \dfrac{e^{-Nc}-1}{e^{-c}-1}$

but This includes The zero order term $(e^{-c})^0 = 1$. subtract it

$$\sum_{k=2}^{N}(e^{-c})^{k-1} = \frac{1-e^{-Nc}}{1-e^{-c}} - \frac{1-e^{-c}}{1-e^{-c}} = \frac{e^{-c}-e^{-Nc}}{1-e^{-c}} \equiv f(c,N)$$

so The $\Sigma_i \Sigma_j \mathrm{cov}_{ij}$ will be something like

$$N + 2f(c,N)$$

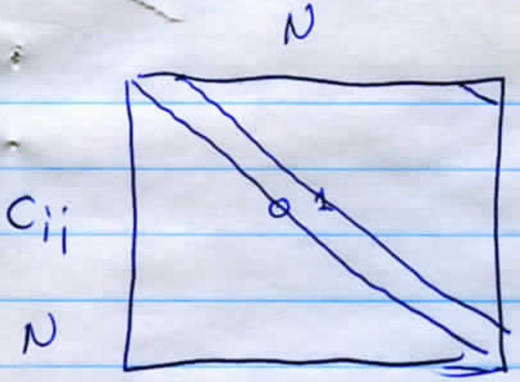
ignoring The fact That all The rows are not quite The same as The first. so The detection condition would be

$$N\,a b > 2\sqrt{N+2f(c,N)} \quad \text{an}$$

$$\frac{1}{2}\frac{N}{\sqrt{N+2f}}\frac{b}{n} > 1$$

but This is approximation.

exact soln



main diagonal $\Sigma = N$

first diagonal $\Sigma = (N-1)\, e^{-c}$

2nd diag $\Sigma = (N-2)\, e^{-2c}$

last diagonal $\Sigma = 1\, e^{-(N-1)c}$

total sum has for $\displaystyle\sum_{j=1}^{N} (N-j+1)\left(e^{-c}\right)^{j-1}$

$$= \underbrace{(N+1)\sum_{j=1}^{N} (e^{-c})^{j-1}}_{\text{Term 1}} - \underbrace{\sum_{j=1}^{N} j\,(e^{-c})^{j-1}}_{\text{term 2}}$$

Term 1 is just from before $(N+1)\dfrac{1-e^{-Nc}}{1-e^{-c}}$

for term 2 see G+R 0.113 with $a=0$ and $r=1$

$$\sum_{k=0}^{N-1} k\, q^k = -\frac{(N-1)\, q^N}{1-q} + \frac{q\,(1-q^{N-1})}{(1-q)^2}$$

The fact that the sum starts at 0 rather than 1 is irrelevant since $k=0$ appears in leading term so

$$\sum_{k=1}^{N} k\, q^k = \frac{-Nq^N}{1-q} + \frac{q\,(1-q^N)}{(1-q)^2} =$$

$$\frac{q\,(1-q^N)}{(1-q)^2} - \frac{Nq^N(1-q)}{(1-q)^2} = \frac{q - q^{N+1} - Nq^N + Nq^{N+1}}{(1-q)^2}$$

$$= \frac{(N-1)q^{N+1} - Nq^{N} + q}{(1-q)^2} = g(c,N) \quad \text{with } q = e^{-c}$$

so $\sum_i \sum_j C_{ij}$ is twice this minus N (since we've counted the main diagonal twice) $\left(2g(c,N) - N\right)$

and detection criteria is

$$\frac{1}{2} \frac{N}{\sqrt{2g - N}} \frac{b}{n} > 1$$