MRN075

Simple Zero-D thermal model of a radioactive, convecting earth. Bill Menke, February 2012.

Heat in mantle $Q = V c_p \rho T = CT$

(assumes that mantle is convectively mixed so it is isothermal)

temperature T

Volume of mantle $V = 4 \pi R^3 / 3$

heat capacity per unit mass = c_p

density ρ

summary constant $C = V c_p \rho$

also define T_0 as initial temperature of mantle

Heat production of mantle $H = V h \rho \exp(-ct) = H_0 \exp(-ct)$

(assume heat production decays exponentially with time)

Initial heat production per unit mass h

decay rate c

Summary constant $H_0 = V h \rho$

Heat loss through conductive lithosphere $q = k A T / L = K T$

(assumes lithosphere thin enough that equilibration time is short)

thermal conductivity k

thickness of conductive lithosphere = L

area of surface of earth $A = 4 \pi R^2$

summary constant $K = k A / L$

Conservation of energy

rate of increase in heat with time t = heat produced – heat lost through surface

$dQ/dt = H(t) - q$

$dT/dt = (H_0/C) \exp(-ct) - (K/C) T$

$dT/dT + aT = b \exp(-ct)$ with $a = (K/C)$ and $b = (H_0/C)$

Solution of ODE (see http://www.sosmath.com/diffeq/first/lineareq/lineareq.html)

$p(t) = a$ and $r(t) = b \exp(-ct)$

$u(t) = \exp(\int p \, dt) = \exp(at)$

$\int u(t) r(t) dt = \int \exp(at) b \exp(-ct) dt = \int b \exp\{(a-c)t\} \, dt = (b/(a-c))\exp((a-c)t)$

$T(t) = (\int u(t) r(t) dt + C) / u(t) = [\{b/(a-c)\} \exp\{(a-c)t\} + C] \exp(-at)$

$T(t=0) = \{b/(a-c)\} + C = T_0$

$C = T_0 - \{b/(a-c)\}$

$T(t) = [\{b/(a-c)\} \exp\{(a-c)t\} + T_0 - \{b/(a-c)\}] \exp(-at)$

Analysis

Suppose no heat production, then b=0 and $T(t) = T_0 \exp(-at)$, so temperature declines exponentially with time at a rate a=(K/C).

Suppose non-zero heat production (b>0) with infinite half life (c=0), then at long time T(t) = b/a; that is, an equilibrium temperature is reached where the mantle heat production is in equilibrium with the conductive loss through the lithosphere.

Numerical Values

R = 6400000 m

$V = 4 \pi R^3 / 3$ = 1.04686E+21 m^3

c_p= 950 J/kg-K

ρ = 5000 kg/m^3

$C = V c_p \rho$ = 4.9726E+27 J/K

h (today) = 4E-12 J/s-kg

h(0) = 2E-11 J/s-kg (assume factor of 5 higher than today)

$H_0 = V h \rho$ = 1.1E+14 J/s

k = 2.7 J/s-m-K

L = 1E5 m (100 km thick lithosphere)

$A = 4 \pi R^2$ = 5.14E+14 m2

K = k A / L = 1.3E+10 J/s-K

a= (K/C)= 2.7E-18 1/s

1/a = 1.2E+10 yrs

so characteristic cooling time in absence or radioactivity is very long, 10 billion years

b = (H_0/C)= 3.15789E-14 K/s

b/a = 1.1667E+04 K

so if radioactivity didn't decay with time, mantle would reach 10,000 K and melt

Quasi-Kelvin's Age of the earth, time it takes heat flow to decline to given value f J/m2-s in the absence of radioactive heating

T_0 = (1350 + 273 + 150) = 1773 K

f= 0.03 J/m2-s

f= k (T_0 /L) exp(-at) = f0 exp(-at) with f0 = k (T_0 /L) = 0.045 J/m2-s

t = -ln(f/f0) /a = 5.6E9 yrs

So convection alone can give a long age of the earth; radioactivity not needed