weighted mean Proof $\bar{y} = \sum_i y_i \sigma_{y_i}^{-2} / \sum_i \sigma_{y_i}^{-2}$

$$E = \sum_i (y_i - \bar{y})^2 / \sigma_i^2$$

$$\frac{\partial E}{\partial \bar{y}} = 0 = \sum_i (-2)(y_i - \bar{y}) / \sigma_i^2$$

$$0 = \sum_i \frac{y_i}{\sigma_i^2} - \bar{y} \sum_i \sigma_i^{-2}$$

$$\bar{y} = \sum_i \frac{y_i}{\sigma_i^2} \Big/ \sum_i \frac{1}{\sigma_i^2}$$

Proof max $r = 1$.

let $r = \sum_i x_i y_i (\sum_i x_i^2)^{-1/2} (\sum_i y_i^2)^{-1/2}$

max r w.r.t. y_i w.c. $\sum y_i = 0$ $\sum y_i^2 = 1$ also $\sum x_i = 0$

$$\phi = \sum x_i y_i (\sum x_i^2)^{-1/2} (\sum y_i^2)^{-1/2} + \lambda_1 \sum_i y_i + \lambda_2 (\sum_i y_i^2 - 1)$$

$$\frac{\partial \phi}{\partial y_k} = x_k (\sum_i x_i^2)^{-1/2} (\sum_i y_i^2)^{-1/2} - \sum_i x_i y_i (\sum_i x_i^2)^{-1/2} y_k (\sum y_i^2)^{-3/2} + \lambda_1 + 2\lambda_2 y_k = 0$$

OR $\quad x_k (\sum_i x_i^2)^{1/2} - \sum_i x_i y_i (\sum_i x_i^2)^{-1/2} y_k + \lambda_1 + 2\lambda_2 y_k = 0$

$\sum_k$: $\quad 0 - 0 + \lambda_1 + 0 = 0 \qquad \lambda_1 = 0$

$$\sum_k y_k \quad \frac{\sum_i x_i y_i}{(\sum_i x_i^2)^{1/2}} - \frac{\sum_i (x_i y_i)}{(\sum x_i^2)^{-1/2}} + 0 + 2\lambda_2 \sum_i y_i^2 = 0 \qquad \lambda_2 = 0$$

so $\quad x_k - (\sum x_i y_i)\, y_k = 0 \quad$ or $\quad y_k = c\, x_k$

$$\sum x_i y_i = c \sum x_i^2$$

$$x_k = c^2 \sum x_i^2\, x_k$$

$$c = (\sum x_i^2)^{-1/2}$$

and $\quad y_k = \frac{x_k}{(\sum x_i^2)^{1/2}} \qquad r = \frac{\sum x_i^2}{(\sum x_i^2)^{1/2} (\sum x_i^2)^{1/2}} = 1$

$$\begin{vmatrix} \sigma_x^2 & \sigma_x\sigma_y\rho \\ \sigma_x\sigma_y\rho & \sigma_y^2 \end{vmatrix} = \sigma_x^2\sigma_y^2 - \sigma_x^2\sigma_y^2\rho^2$$

$$= \sigma_x^2\sigma_y^2(1-\rho^2)$$

$\sigma_x^2 \cdot$

$$\frac{\begin{bmatrix} \sigma_y^2 & -\sigma_x\sigma_y\rho \\ \sigma_x\sigma_y\rho & \sigma_x^2 \end{bmatrix}}{\sigma_x^2\sigma_y^2(1-\rho^2)}$$

<u>Proof</u> $r = 1$ with $r = \left(\sum \frac{x_i}{\sigma_{x_i}}\frac{y_i}{\sigma_{y_i}}\right)\left(\sum_i \frac{x_i^2}{\sigma_{x_i}^2}\right)^{-1/2}\left(\sum_i \frac{y_i}{\sigma_{y_i}^2}\right)^{-\frac{1}{2}}$

max r w.r.t. y_k w.c. $\sum \frac{y_i}{\sigma_{y_i}^2} = 0$ $\quad \sum \frac{y_i^2}{\sigma_{y_i}^2} = 1$

also $\sum \frac{x_i}{\sigma_{x_i}^2} = 0$

$\phi =$

$$\phi = \left(\sum \frac{x_i}{\sigma_{x_i}}\frac{y_i}{\sigma_{y_i}}\right)\left(\sum \frac{x_i^2}{\sigma_{x_i}^2}\right)^{-1/2}\left(\sum \frac{y_i^2}{\sigma_{y_i}^2}\right)^{-1/2} + \lambda_1 \sum \frac{y_i}{\sigma_{y_i}^2} + \lambda_2\left(\sum \frac{y_i^2}{\sigma_{y_i}^2} - 1\right)$$

$$\frac{\partial \phi}{\partial y_k}: \quad \frac{x_k}{\sigma_{x_k}}\frac{1}{\sigma_{y_k}}\left(\sum \frac{x_i^2}{\sigma_{x_i}^2}\right)^{-1/2}\left(\sum \frac{y_i^2}{\sigma_{y_i}^2}\right)^{-1/2}$$

$$- \left(\sum \frac{x_i}{\sigma_{x_i}}\frac{y_i}{\sigma_{y_i}}\right)\left(\sum \frac{x_i^2}{\sigma_{x_i}^2}\right)^{-1/2}\frac{y_k}{\sigma_{y_k}^2}\left(\sum \frac{y_i^2}{\sigma_{y_i}^2}\right)^{-3/2} + \lambda_1 \frac{1}{\sigma_{y_k}^2} + 2\lambda_2 \frac{y_k}{\sigma_{y_k}^2} = 0$$

$$0 := \frac{x_k}{\sigma_{x_k}}\frac{1}{\sigma_{y_k}}\left(\sum \frac{x_i^2}{\sigma_{x_i}^2}\right)^{-1/2} - \sum \frac{x_i y_i}{\sigma_{x_i}\sigma_{y_i}}\left(\sum \frac{x_i^2}{\sigma_{x_i}^2}\right)^{-1/2}\frac{y_k}{\sigma_{y_k}^2} + \frac{\lambda_1}{\sigma_{y_k}^2} + \frac{\lambda_2 y_k}{\sigma_{y_k}^2}$$

$$\sum_k: \left(\sum \frac{x_k}{\sigma_{x_k}\sigma_{y_k}}\right)\left(\sum \frac{x_i^2}{\sigma_{x_i}^2}\right)^{-1/2} - 0 + \lambda_1 \sum \sigma_{y_k}^{-2} + 0 = 0$$

$$\lambda_1 = -\sum \frac{x_k}{\sigma_{x_k}\sigma_{y_k}}\left(\sum \frac{x_i^2}{\sigma_{x_i}^2}\right)^{-1/2}\left(\sum \sigma_{y_k}^{-2}\right)^{-1}$$

$$\sum_k y_k: \left(\sum \frac{xy}{\sigma_x\sigma_y}\right)\left(\sum \frac{x^2}{\sigma_x^2}\right)^{-1/2} - \left(\sum \frac{x_i y_i}{\sigma_x\sigma_y}\right)\left(\sum \frac{x_i^2}{\sigma_{x_i}^2}\right) + 0 + \lambda_2 = 0$$

$$\lambda_2 = 0$$

so

$$\frac{x_k}{\sigma_x\sigma_y}\left(\sum x^2/\sigma_x^2\right)^{-1/2} - \left(\sum \frac{x_i y_i}{\sigma_x\sigma_y}\right)\left(\sum \frac{x^2}{\sigma_x^2}\right)^{-1/2}\frac{y_k}{\sigma_{y_k}^2} - \left(\sum \frac{x_k}{\sigma_x\sigma_y}\right)\left(\sum \frac{x^2}{\sigma_x^2}\right)^{-1}\left(\sum \sigma_y^{-2}\right)^{-1}\frac{1}{\sigma_{y_k}^2}$$

$r = \left(\sum \frac{x}{\sigma_x}\frac{y}{\sigma_y}\right)\left(\sum \frac{x^2}{\sigma_x^2}\right)^{-1/2}\left(\sum \frac{y^2}{\sigma_y^2}\right)^{-1/2}$ $\quad \sum y_i/\sigma_{y_i} = 0$

$\phi = \left(\sum \frac{x}{\sigma_x}\frac{y}{\sigma_y}\right)\left(\sum \frac{x^2}{\sigma_x^2}\right)^{-1/2}\left(\sum \frac{y^2}{\sigma_y^2}\right)^{-1/2} + \lambda_1\left(\sum \frac{y}{\sigma_y^2}\right) + \lambda_2\left(\sum \frac{y^2}{\sigma_y} - 1\right)$ with $\sum y_i^2/\sigma_{y_i}^2 = 1$

$\sum x_i/\sigma_{x_i} = 0$

$\frac{\partial\phi}{\partial y} = \frac{x}{\sigma_x\sigma_y}\left(\sum \frac{x^2}{\sigma_x^2}\right)^{-1/2} - \left(\sum \frac{x}{\sigma_x}\frac{y}{\sigma_y}\right)\left(\sum \frac{x^2}{\sigma_x^2}\right)^{-1/2}\frac{y}{\sigma_y^2} + \lambda_1\frac{1}{\sigma_y^2} + 2\lambda_2\frac{y}{\sigma_y^2} = 0$

$\sum$: $\left(\sum \frac{x}{\sigma_x\sigma_y}\right)\left(\sum \frac{x^2}{\sigma_x^2}\right)^{-1/2} - \left(\sum \frac{xy}{\sigma_x\sigma_y}\right) 0 \left(\sum \frac{x^2}{\sigma_x^2}\right)^{-1/2} + \lambda_1\left(\sum \frac{1}{\sigma_y^2}\right) + 0 = 0$ $\quad \Rightarrow \quad \lambda_1 = -\frac{\sum \frac{x}{\sigma_x\sigma_y}\sum\frac{x^2}{\sigma_x^2}}{\sum \frac{1}{\sigma_y^2}}$

$\sum y$ $\left(\sum \frac{xy}{\sigma_x\sigma_y}\right)\left(\sum \frac{x^2}{\sigma_x^2}\right)^{-1/2} - \left(\sum \frac{xy}{\sigma_x\sigma_y}\right)\left(\sum \frac{x^2}{\sigma_x^2}\right)^{-1/2} + 0 + 2\lambda_2 \quad \Rightarrow \quad \lambda_2 = 0$

$\frac{x}{\sigma_x\sigma_y}\left(\sum \frac{x^2}{\sigma_x^2}\right)^{-1/2} - \left(\sum \frac{xy}{\sigma_x\sigma_y}\right)\left(\sum \frac{x^2}{\sigma_x^2}\right)^{-1/2}\frac{y}{\sigma_y^2} - \frac{\sum \frac{x}{\sigma_x\sigma_y}\sum\frac{x^2}{\sigma_x^2}}{\sum 1/\sigma_y^2}\frac{1}{\sigma_y^2} = 0$

x