MRN095

Valid Forms of Ray Eqn

$$\nabla T = \frac{\hat{t}}{c} \qquad (A)$$

$$(\nabla T)^2 = \frac{1}{c^2} \qquad (B)$$

$$\nabla T \cdot \hat{t} = \frac{dT}{ds} = \frac{1}{c} \qquad (C)$$

$$\frac{d}{ds} c^{-1} \frac{dx}{ds} = \nabla c^{-1} \qquad (D)$$

$$\frac{d}{ds} \hat{t} = \hat{t} \times (c \nabla c^{-1} \times \hat{t}) \qquad (E)$$

ray viewpoint $\underline{x}(T, p, q)$ — p, q coords in surface of wavefront

T, p, q const orthog. curv. coords

wavefront viewpoint $\underline{r} = [T, p, q]^T$

$$\underline{r}(\underline{x}) = \text{constant}$$

general relationship $\frac{\partial r_i}{\partial x_k} \frac{\partial x_k}{\partial r_j} = \delta_{ij} \qquad \frac{\partial x_i}{\partial r_k} \frac{\partial r_k}{\partial x_j} = \delta_{ij}$

so for $r_1 = T$

$$\nabla T \cdot \frac{\partial \underline{x}}{\partial T} = 1 \qquad (F)$$

Proof $\frac{d\underline{x}}{dT} = c\hat{t}$

~~dt~~

[illegible]

dot with $\frac{dT}{d\underline{x}}$

$$\frac{dT}{d\underline{x}} \cdot \frac{d\underline{x}}{dT} = c\,\hat{t}\cdot\frac{dT}{d\underline{x}}$$

employ F

$$1 = c\,\hat{t}\cdot\nabla T$$

$$\frac{1}{c} = \hat{t}\cdot\nabla T$$

which is C

manip of ray eqn

$$\frac{d}{ds}\hat{t} = \hat{t} \times \left(c \nabla c^{-1} \times \hat{t} \right)$$

$$\nabla c^{-1} = -c^{-2} \nabla c$$

$$\frac{d}{ds} = \frac{1}{c}\frac{d}{dT} \qquad c \nabla c^{-1} = -c^{-1} \nabla c$$

$$\frac{d}{dT}\hat{t} = \hat{t} \times \left(\hat{t} \times \nabla c \right)$$

$$\hat{t} \times \frac{d\hat{t}}{dT} = \underset{a}{\hat{t}} \times \left(\underset{b}{\hat{t}} \times \underset{c}{[\hat{t} \times \nabla c]} \right) \qquad a \times (b \times c) = (a \cdot c)\, b - (a \cdot b)\, c$$

$$= \cancelto{0}{\hat{t} \cdot [\hat{t} \times \nabla c]}\, \hat{t} - (\hat{t} \cdot \hat{t})(\hat{t} \times \nabla c)$$

$$= -\hat{t} \times \nabla c$$

$$= \nabla c \times \hat{t}$$

Intuitive Eqns

$$\frac{dx}{dT} = c\hat{t}$$ proved to be a valid form of ray eqn. (G)

$$\hat{t} = \frac{\partial x}{\partial p} \times \frac{\partial x}{\partial q}$$ simple geometrical statement. That $\hat{t}$ normal to surface. (H)

Velocity gradient

From (6)

$$\frac{\partial x}{\partial T} = c\hat{t}$$

$$\frac{d}{dp}; \quad \frac{\partial^2 x}{\partial T \partial p} = \frac{\partial}{\partial p}(c\hat{t}) = \frac{\partial c}{\partial p}\hat{t} + c\frac{\partial \hat{t}}{\partial p}$$

$$\hat{t}\cdot; \quad \hat{t}\cdot\frac{\partial^2 x}{\partial T \partial p} = \frac{\partial c}{\partial p} + c\left(\frac{\partial \hat{t}}{\partial p}\cdot\hat{t}\right)$$

zero since $\frac{\partial \hat{t}}{\partial p} \perp \hat{t}$

$$\frac{\partial c}{\partial p} = \hat{t}\cdot\frac{\partial^2 x}{\partial T \partial p}$$

$$\frac{\partial c}{\partial q} = \hat{t}\cdot\frac{\partial^2 x}{\partial T \partial q}$$

Proof That Ray eqn can be recovered by combining (G) and (H)

$$\hat{t} = \frac{\partial x}{\partial p} \times \frac{\partial x}{\partial q}$$

$\frac{d}{d\tau}$: $$\frac{d\hat{t}}{\partial \tau} = \frac{\partial^2 x}{\partial p \partial \tau} \times \frac{\partial x}{\partial q} - \frac{\partial^2 x}{\partial q \partial \tau} \times \frac{\partial x}{\partial p}$$

$\hat{t}\times$: $$\hat{t} \times \frac{d\hat{t}}{\partial \tau} = \underset{a}{\hat{t}} \times \left(\underset{b}{\frac{\partial^2 x}{\partial p \partial \tau}} \times \underset{c}{\frac{\partial x}{\partial q}} \right) - \hat{t} \times \left(\frac{\partial^2 x}{\partial q \partial \tau} \times \frac{\partial x}{\partial p} \right)$$ $a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$

$$= \cancel{\left(\hat{t} \cdot \frac{\partial x}{\partial q} \right) \frac{\partial^2 x}{\partial p \partial \tau}} - \hat{t} \cdot \frac{\partial^2 x}{\partial p \partial \tau} \frac{\partial x}{\partial q} - \cancel{\left(\hat{t} \cdot \frac{\partial x}{\partial p} \right) \frac{\partial^2 x}{\partial q \partial \tau}} + \hat{t} \cdot \frac{\partial^2 x}{\partial q \partial \tau} \frac{\partial x}{\partial p}$$

$0 \quad \hat{t} \perp \frac{\partial x}{\partial q}$ $\qquad 0 \quad \hat{t} \perp \frac{\partial x}{\partial p}$

$$= \frac{\partial c}{\partial q} \frac{\partial x}{\partial p} - \frac{\partial c}{\partial p} \frac{\partial x}{\partial q}$$

$$\boxed{\hat{t} \times \frac{d\hat{t}}{\partial \tau} = \frac{\partial c}{\partial q} \frac{\partial x}{\partial p} - \frac{\partial c}{\partial p} \frac{\partial x}{\partial q}}$$

by $\frac{\partial c}{\partial q} = \frac{\partial c}{\partial x_i} \frac{\partial x_i}{\partial q} = \nabla c \cdot \hat{q}$; $\frac{\partial c}{\partial p} = \frac{\partial c}{\partial x_i} \frac{\partial x_i}{\partial p} = \nabla c \cdot \hat{p}$

$$\hat{t} \times \frac{d\hat{t}}{\partial \tau} = (\nabla c \cdot \hat{q}) \hat{p} - (\nabla c \cdot \hat{p}) \hat{q}$$

a c b $\qquad$ a b c

$$= \nabla c \times (\hat{q} \times \hat{p})$$

$$= \nabla c \times \hat{t}$$