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Gauge Invariance

given a set of scalar fields ϕ

Suppose observables are invariant under a rotation G of the fields ... $\phi' \to G\phi$

note That, The Lagrangian Density
$\frac{1}{2}(\nabla\phi)^T(\nabla\phi) - \frac{1}{2}m^2\phi^T\phi$
is invariant, as long as G is constant in space.

But suppose we want a Lagrangian to be invariant under a local (spacially varying) rotation $G(x)$. (The orthogonal Group $O(n)$)

Then define The covariant derivative $D = (\nabla + gA)$ so that

$$(\nabla + gA')\phi' \longrightarrow G(\nabla + gA)\phi$$

with $\nabla' \to \nabla$ and $A' \to GAG^{-1} + \frac{1}{g}G(\nabla G^{-1})$

proof $(\nabla' + gA')\phi' =$

$$(\nabla'\phi' + gGAG^{-1}\phi' + G(\nabla G^{-1})\phi' =$$
$$\nabla(G\phi) + gGAG^{-1}G\phi + G(\nabla G^{-1})G\phi =$$
$$G(\nabla + gA)\phi + (\nabla G)\phi - GG^{-1}\nabla G G^{-1}\phi =$$
$$G(\nabla + gA)\phi + (\nabla G)\phi - (\nabla G)\phi =$$
$$G(\nabla + gA)\phi$$

note $\nabla G^{-1} = -G^{-1}(\nabla G)G^{-1}$ from $\nabla(GG^{-1}) = 0$

A is called The "gauge field" and g is called The coupling constant.

Then one postulates a new, locally invariant Lagrangian

$$\frac{1}{2}(D\phi)^T(D\phi) + \frac{m}{2}\phi^T\phi$$

The difference between this Langrangian and the original one is

$$\Delta L = \frac{g}{2}\phi^T A^T \nabla\phi + \frac{g}{2}(\nabla\phi)^T A\phi + \frac{g^2}{2}(A\phi)^T(A\phi)$$

which can be intepreted as couplings between the fields, eg, terms like $A_{12}\phi_1\phi_2$.

The gauge field A is not uniquely determined by the requirement of the transformation law, but is constrained by it in a way I don't completely understand.

Wikipedia says, "The guage field is an element of the Lie algebra, and therefore can be expanded as $A(x) = \sum A^\alpha(x) T^\alpha$"
where "T^α matrices are generators of the group $SO(n)$. ($SO(n)$ is the subgroup of $O(n)$ with $\det = 1$)

from some reading I've done, it looks like the generators of the representation of the rotation group $SO(n)$ are just the derivatives

$$T^\alpha = -i \left.\frac{\partial G}{\partial\theta_i}\right|_{\theta=0} \quad w/\ \theta_i = \text{euler angle.}$$

and so are just 3 very simple matrices like $\begin{pmatrix} 0&0&0 \\ 0&0&-i \\ 0&i&0 \end{pmatrix}$
(they are generators in the sense that a general rotation is $\exp(i T^\alpha n_\alpha)$)

The Lie algebra can be used to derive a Lagrangian for A: I
$F_{ij} = D_i D_j - D_j D_i$ Then $L = \frac{1}{2g^2} Tr(F_{ij}F^{ij})$.
Note $F' = GFG^{-1}$, so Lagrangian is Gauge Invariant.

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Lie group G eg. Group of all possible rotations of a 3D object

elements R depending upon n independent parameters

$R(\underline{\xi}) = R(\xi_1, \xi_2 \cdots \xi_n)$ n = dimension of Group

$R = I$ is identity element

$R(\underline{\varepsilon}) \approx I + \sum \varepsilon_i S_i$ with $S_i = \frac{1}{i} \frac{\partial}{\partial \xi_i} R(\underline{\xi}) \Big|_{\underline{\xi}=0}$

S_i "generators"

commutator $[S_i, S_j] = S_i S_j - S_j S_i$ measures non-communtivity of S's

since $R_i R_j \stackrel{?}{=} R_j R_i$ same as $R_i^{-1} R_j^{-1} R_i R_j \stackrel{?}{=} I$

and $R_i^{-1} R_j^{-1} R_i R_j = I + \varepsilon_i \varepsilon_j [S_i, S_j]$

commutator can be expressed as linear comb of Generators

$$[S_i, S_j] = \sum_k^n c_{ij}^k S_k$$

structureconstant of Lie Algebra, a defining relation

since $[\,,\,]$ antisymmetric

Then $c_{ij}^k = -c_{ji}^k$

Jacobi Identity $[[S_i, S_j], S_k] + [[S_j, S_k], S_i] + [[S_k, S_i], S_j] = 0$

Example $SO(3)$ group of rotations in 3D

O = orthogonal (real, $A^T = A^{-1}$)

S = simple $\det A = 1$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_1 & \sin\phi_1 \\ 0 & -\sin\phi_1 & \cos\phi_1 \end{pmatrix} \begin{pmatrix} \cos\phi_2 & 0 & -\sin\phi_2 \\ 0 & 1 & 0 \\ \sin\phi_2 & 0 & \cos\phi_2 \end{pmatrix} \begin{pmatrix} \cos\phi_3 & \sin\phi_3 & 0 \\ -\sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_i = -i \frac{dR}{\partial \phi_i}\Big|_{\underline{\phi}=0} \quad S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example $SU(2)$ U = unary ($A^{-1} = A^{H}$) simple $\det A = 1$

$$R = \begin{pmatrix} \exp(i\xi)\cos\eta & \exp(i\zeta)\sin\eta \\ -\exp(-i\zeta)\sin\eta & \exp(-i\xi)\cos\eta \end{pmatrix}$$

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$SU(3)$ 9 elements x 2 real/img parts = 18 minus

$A^T A = I$: 9 real $\det(A) = 1$: 1 $18 - 9 - 1 = 8$ parameters.

(4) getting back to gauge Theory ...

The step of expanding $A(x)$ in The generators T of The Lie Algebra is really important: $A(x) = \sum_{\alpha}^{n} A_n(x) T^n$ since it is The way That The group symmetry is explicitly introduced into The Lagrangian That is, Thru The covariant derivative

$$D_\mu = \partial_\mu + g \sum_{\alpha}^{n} A_n(x) T^n$$

note Then That There are n gauge fields, A_α $\alpha = 1, N$ one for each generator, which is to say, as many as The order of The group.

QED local invariance under $U(1)$ (phase change)
1 gauge field (electromagnetic field)
1 generator: i → photon

Electroweak $SU(2)$ and $U(1)$
→ 3 gauge fields W^+, W^-, Z bosons

Standard model $SU(3)$ and $SU(2)$ and $U(1)$

QCD $SU(3)$ → 8 gauge fields
→ gluons

note Yang-Mills Lagrangian $\mathcal{L}_{YM}$

lie algebra: $[T^a, T^b] = i f^{abc} T^c$

then $D_\mu = \partial_\mu - i g A_\mu^\alpha T^\alpha \qquad A_\mu = T^\alpha A_\mu^\alpha$

$$F_{\mu\nu}^a = \partial_\mu A_\mu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

$$\mathcal{L}_{YM} = -\frac{1}{4} Tr\left(F_a^{\mu\nu} F_{\mu\nu}^a \right)$$