EQUATIONS OF FLUID FLOW IN non-consolidating rigid sediment

<u>darcy's law</u> velocity = − permeability · pressure gradient

$$\underline{V} = -\frac{k}{\rho_{H_2O}\, g\, n}\nabla P = K\nabla P$$

<u>compressibility</u>

$$\beta = -\frac{1}{v}\frac{\partial v}{\partial P} = \frac{1}{\rho}\frac{\partial \rho}{\partial P}$$

v = volume
ρ = density

<u>mass conservation</u>

$$\frac{\partial(\rho n)}{\partial t} = -\nabla\cdot\{n\rho\underline{V}\}$$

n = porosity

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} = -\nabla\cdot\{\rho\underline{V}\}$$

iff n = constant

<u>note</u>

a) $\nabla\cdot\{\rho\underline{V}\} = \rho\,\nabla\cdot\underline{V} + \underline{V}\cdot\nabla\rho$

b) $-\frac{1}{\rho}\frac{\partial \rho}{\partial t} = -\frac{1}{\rho}\frac{\partial\rho}{\partial P}\frac{\partial P}{\partial t} = -\beta\frac{\partial P}{\partial t}$

c) $\frac{\partial\rho}{\partial P} = \frac{\partial\rho}{\partial x}\frac{\partial x}{\partial P} = \beta\rho$ so $\frac{\nabla\rho}{\rho} = \beta\nabla P$

from mass conservation

$$-\frac{\partial\rho}{\partial t} = \rho\,\nabla\cdot\underline{V} + \underline{V}\cdot\nabla\rho$$

dividing by ρ

$$-\frac{1}{\rho}\frac{\partial\rho}{\partial t} = \nabla\cdot\underline{V} + \underline{V}\cdot\frac{\nabla\rho}{\rho}$$

using notes b) and c)

$$-\beta\frac{\partial P}{\partial t} = \nabla\cdot\underline{V} + \underline{V}\cdot\beta\nabla P$$

inserting darcy's law

$$+\beta\frac{\partial P}{\partial t} = \nabla\cdot(K\nabla P) + K\,\nabla P\cdot\beta\nabla P$$

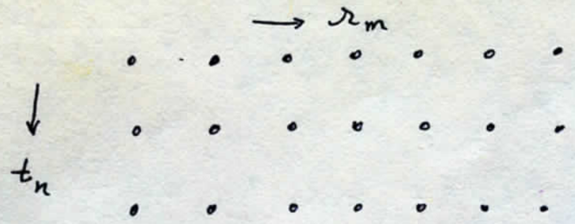
assuming K constant

$$\beta\frac{\partial P}{\partial t} = K\,\nabla^2 P + K\beta(\nabla P)^2$$

in cylindrical coordinates this is $\nabla P \rightarrow \frac{\partial P}{\partial r}$ & $\nabla^2 P = \left(\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r}\right)$

$$\beta\frac{\partial P}{\partial t} = K\frac{\partial^2 P}{\partial r^2} + \frac{K}{r}\frac{\partial P}{\partial r} + K\beta\left(\frac{\partial P}{\partial r}\right)^2$$

finite difference scheme



center derivatives on $(m, n+\frac{1}{2})$

$$\frac{\partial P}{\partial t} = \frac{P_m^{n+1} - P_m^n}{\Delta t}$$

$$\frac{\partial^2 P}{\partial r^2} = \frac{1}{2\Delta r^2}\left(P_{m+1}^{n+1} - 2P_m^{n+1} + P_{m-1}^{n+1} + P_{m+1}^n - 2P_m^n + P_{m-1}^n\right)$$

$$\frac{\partial P}{\partial r} = \frac{1}{4\Delta r}\left(P_{m+1}^{n+1} - P_{m-1}^{n+1} + P_{m+1}^n - P_{m-1}^n\right)$$

EQUATION BECOMES

$$\frac{\beta}{\Delta t}\left(P_m^{n+1} - P_m^n\right) = \frac{K}{2\Delta r^2}\left(P_{m+1}^{n+1} - 2P_m^{n+1} + P_{m-1}^{n+1} + P_{m+1}^n - 2P_m^n + P_{m-1}^n\right)$$

$$+ \frac{K}{4\Delta r\, r_m}\left(P_{m+1}^{n+1} - P_{m-1}^{n+1} + P_{m+1}^n - P_{m-1}^n\right)$$

$$+ \frac{K\beta}{16\Delta r^2}\left(P_{m+1}^{\prime\, n+1} - P_{m-1}^{\prime\, n+1} + P_{m+1}^n - P_{m-1}^n\right)^2$$

rearrange:

$$\left(-\frac{K}{2\Delta r^2} + \frac{K}{4\Delta r\, r_m}\right)P_{m-1}^{n+1} + \left(\frac{\beta}{\Delta t} + \frac{K}{\Delta r^2}\right)P_m^{n+1} + \left(-\frac{K}{2\Delta r^2} - \frac{K}{4\Delta r\, r_m}\right)P_{m+1}^{n+1}$$

$$= \left(\frac{K}{2\Delta r^2} - \frac{K}{4\Delta r\, r_m}\right)P_{m-1}^n + \left(\frac{\beta}{\Delta t} - \frac{K}{\Delta r^2}\right)P_m^n + \left(\frac{K}{2\Delta r^2} + \frac{K}{4\Delta r\, r_m}\right)P_{m+1}^n$$

$$+ \frac{K\beta}{16\Delta r^2}\left(P_{m+1}^{\prime\, n+1} - P_{m-1}^{\prime\, n+1} + P_{m+1}^n - P_{m-1}^n\right)^2$$

solution of non-linear fluid flow equation in media with constant porosity and permeability.

BASIC equation $\beta \frac{\partial P}{\partial t} = K \frac{\partial^2 P}{\partial x^2} + \beta K \left(\frac{\partial P}{\partial x}\right)^2$

β = compressability

$K = \frac{k}{\eta \rho g}$

TIME INDEPENDENT ASSUMPTION $0 = \frac{d^2P}{dx^2} + \beta \left(\frac{dP}{dx}\right)^2$

solve by writing $y = \frac{dP}{dx}$ $\frac{dy}{y^2} = -\beta dx$

general solution is $P(x) = \frac{1}{\beta} \ln(x + C_0) + C_1$ C_0, C_1 constants.

suppose boundary conditions $P(0) = 1$ $P(1) = 2$ Then

$$C_0 = (e^\beta - 1)^{-1}$$

$$C_1 = 1 + \frac{1}{\beta} \ln(e^\beta - 1)$$

since $V = -K \frac{dP}{dx}$ we have

$$P(x) = \frac{1}{\beta} \ln\left(x(e^\beta - 1) + 1\right) + 1$$

$$V(x) = -\frac{K}{\beta} \frac{(e^\beta - 1)}{x(e^\beta - 1) + 1}$$

assume $\beta \ll 1$ Then since $e^x - 1 \approx (x + \frac{x^2}{2})$ and $\ln(x+1) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3$ we have:

$$P(x) \approx 1 + (1 + \tfrac{\beta}{2})x - \tfrac{\beta}{2}x^2 \qquad O(\beta)$$

$$V(x) \approx -K\left[(1 + \tfrac{\beta}{2}) - \beta x\right] \qquad O(\beta)$$