definition of ϕ $\qquad \phi = -\log_2\left(\frac{2r}{d_s}\right)$ $\qquad r$ = radius of particle
d_s = standard diameter
= 1 mm

$$\phi = -1 - \frac{\ln r/d_s}{\ln 2} \qquad \ln = \log_e$$

2. size distribution function $\qquad p(\phi)\,d\phi$ = volume fraction of particles in size range ϕ to $\phi + d\phi$

3. distribution in r $\qquad f(r)\,dr = -p(\phi(r))\,\frac{\partial\phi}{\partial r}\,dr$

where $\frac{\partial\phi}{\partial r} = -\frac{1}{r\ln 2}$

4. Total fraction is one $\qquad \int_{-\infty}^{\infty} p(\phi)\,d\phi = \int_0^{\infty} f(r)\,dr = 1$

5. surface area, let porosity be n. The volume of sediment grains is $(1-n)$. The volume fraction with radius near r is $(1-n)f(r)\,dr$. These have surface area $\frac{3}{r}(1-n)f(r)\,dr$ so that surface area per unit volume of sediment is

$$A = 3(1-n)\int_0^{\infty} \frac{1}{r} f(r)\,dr$$

6. volume fraction has normal distribution in ϕ

$$p(\phi)\,d\phi = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{(\phi-\phi_0)^2}{2\sigma^2}\right\}d\phi$$

7. integrating: $A = \frac{3(1-n)}{\sqrt{2\pi}\,\sigma \ln 2} \int_0^\infty r^{-2} \exp\left(-\frac{(\phi(r)-\phi_0)^2}{2\sigma^2}\right) dr$

since $\phi = -1 - \frac{\ln r/d_s}{\ln 2}$ $\quad (\phi-\phi_0)^2 = \frac{1}{(\ln 2)^2}\left(\ln\frac{r}{d_s} - \ln\frac{r_0}{d_s}\right)^2$

$$A = \frac{3(1-n)}{\sqrt{2\pi}\,\sigma \ln 2} \int_0^\infty r^{-2} \exp\left\{-\frac{\left(\ln\frac{r}{d_s} - \ln\frac{r_0}{d_s}\right)^2}{2\sigma^2(\ln 2)^2}\right\} dr$$

let $a = \frac{3(1-n)}{\sqrt{2\pi}\,\sigma \ln 2}$ $\quad b = \ln\frac{r_0}{d_s}$ $\quad c = 2\sigma^2(\ln 2)^2$

$$A = a\int_0^\infty r^{-2} \exp\left\{-\frac{\left(\ln\frac{r}{d_s} - b\right)^2}{c}\right\} dr$$

8. change of variables $\quad y = \ln\frac{r}{d_s} \quad r = d_s e^y \quad \frac{dr}{dy} = d_s e^y$

$$A = a\int_{-\infty}^\infty d_s^{-2} e^{-2y} \exp\left\{-\frac{(y-b)^2}{c}\right\} d_s e^y \, dy$$

$$= \frac{a}{d_s}\int_{-\infty}^\infty \exp\left\{-\frac{(y-b)^2}{c} - y\right\} dy$$

$$= \frac{a}{d_s}\int_{-\infty}^\infty \exp\left\{-\frac{y^2}{c} - \frac{b^2}{c} + \frac{2by}{c} - y\right\} dy$$

$$= \frac{a}{d_s} e^{-\frac{b^2}{c}} \int_{-\infty}^\infty \exp\left\{-\frac{y^2}{c} + \left(\frac{2b}{c} - 1\right) y\right\} dy$$

This integral known — Gradshteyn & Ryzhik give (3.323.2)

$$\int_{-\infty}^{\infty} \exp\left\{ -p^2x^2 \pm qx \right\} dx = \exp\left\{ \frac{q^2}{4p^2} \right\} \frac{\sqrt{\pi}}{p} \quad \text{if } p>0$$

let $p^2 = \frac{1}{c}$ $q = \left(\frac{2b}{c}-1\right)$ Therefore

$$A = \frac{a}{ds}\, e^{-\frac{b^2}{c^2}} \exp\left\{ \frac{\left(\frac{2b}{c}-1\right)^2 c}{4} \right\} \sqrt{\pi}\sqrt{c}$$

$$A = \frac{3(1-n)}{\cancel{\sqrt{2}}\,\cancel{\sqrt{\pi}}\,\cancel{\sigma \ln 2}} \frac{1}{ds} \cancel{\sqrt{2}}\,\cancel{\sqrt{\pi}}\,\cancel{\sigma \ln 2} \exp\left\{ \frac{\left(\frac{2b}{c}-1\right)^2 c}{4} - \frac{b^2}{c} \right\}$$

$$\frac{\left(\frac{2b}{c}-1\right)^2 c}{4} - \frac{b^2}{c} = \frac{\left(\frac{4b^2}{c^2}+1-\frac{4b}{c}\right)c}{4} - \frac{b^2}{c} = \frac{b^2}{c} + \frac{c}{4} - b - \frac{b^2}{c}$$

$$= \frac{c}{4} - b$$

$$A = \frac{3(1-n)}{ds} \exp\left\{ \frac{c}{4} - b \right\} = \frac{3(1-n)}{ds} \exp\left\{ \frac{1}{2}\sigma^2 (\ln 2)^2 - \ln\frac{r_0}{ds} \right\}$$

$$= \boxed{\frac{3(1-n)}{r_0} \exp\left\{ \frac{1}{2}\sigma^2 (\ln 2)^2 \right\}}$$

10. limit $\sigma \to 0$

$$A = \frac{3(1-n)}{ds} \exp\left\{ -\ln\frac{r_0}{ds} \right\}$$

$$= \frac{3(1-n)}{ds} \frac{ds}{r_0}$$

$$= \frac{3(1-n)}{r_0}$$