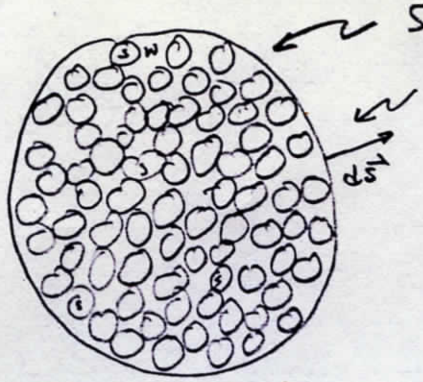
flow of heat in wet sediment



Surface $S'$ inclosing volume $V$ of sediment and water

surface normal

w = water

s = sediment

n = porosity

k = thermal conductivity

$\rho$ = density

c = heat capacity

v = velocity

total heat flux thru $d\vec{s}$ = heat flux thru sediment + heat flux through water

$$\vec{q}_{TOTAL} = q_s + q_w$$

$$\vec{q}_s = -(1-n)\,k_s\,\nabla T$$

$\vec{q}_w$ = conducted thru water + moved by mass flux of water

$$= -n\,k_w\,\nabla T + \rho_w c_w n T \vec{v}$$

$$\vec{q}_{TOTAL} = -\left[(1-n)k_s + n k_w\right]\nabla T + \rho_w c_w\, n T \vec{v}$$

but heat lost from volume $V$ is flux integrated around surface $S'$

$$\iiint_V \left[(1-n)\rho_s c_s + n\rho_w c_w\right]\frac{\partial T}{\partial t} = -\iint_S \vec{q}_{TOTAL}\, d\vec{s}$$

now convert RHS to volume integral by $\iint_S \vec{q}\cdot d\vec{s} = \iiint_V \nabla\cdot q\; dV$ and argue away volume integral

$$\left[(1-n)\rho_s c_s + n\rho_w c_w\right]\frac{\partial T}{\partial t} =$$

$$\nabla\cdot\left\{\left[(1-n)k_s + n k_w\right]\nabla T\right\} - \nabla\cdot\left\{c_w T \rho_w n \vec{v}\right\}$$

let $\left[(1-n)\rho_s c_s + n\rho_w c_w\right] = \rho_b c_b$ ; $\left[(1-n)k_s + n k_w\right] = k_b$

now lets consider $\nabla \cdot \{ c_w \rho_w T n \vec{v} \}$ suppose water has constant heat capacity. Note also $\nabla \cdot (\phi \vec{v}) = \vec{v} \cdot \nabla \phi + \phi \nabla \cdot \vec{v}$.

Then $\nabla \cdot \{ c_w \rho_w T n \vec{v} \} = c_w T \nabla \cdot (\rho_w n \vec{v}) + c_w \rho_w n \vec{v} \cdot \nabla T$

$= c_w \rho_w n \vec{v} \cdot \nabla T$ since we shall assume the mass flux of water past a particular point is conserved.

Switching to 1 dimension:

$$\rho_b c_b \frac{\partial T}{\partial t} + \rho_w c_w n V_z \frac{\partial T}{\partial z} = k_b \frac{\partial^2 T}{\partial z^2} + (k_w - k_s) \frac{\partial n}{\partial z} \frac{\partial T}{\partial z}$$

assumming steady state Temp. distribution

$$\frac{\partial^2 T}{\partial z^2} = \frac{\left[ \rho_w c_w n V_z + (k_s - k_w) \frac{\partial n}{\partial z} \right]}{k_b} \frac{\partial T}{\partial z}$$

suppose $n =$ constant. let $\beta = [\rho_w c_w n V_z] / k_b$

equation is $\frac{\partial^2 T}{\partial z^2} = \beta \frac{\partial T}{\partial z}$ which has general solution

$T(z) = C_1 e^{\beta z} + C_2$. Taking for boundary conditions

$T(z=0) = T_0$ $\quad T(z=L) = T_L$ we have

$$T(z=0) = T_0 = C_1 + C_2$$

$$T(z=L) = T_L = C_1 e^{\beta L} + C_2$$

solve simultaneously to get $C_1$, $C_2$ which gives $T(z)$:

$$C_1 \qquad\qquad C_2$$

$$T(z) = \left[\frac{T_L - T_0}{e^{\beta L} - 1}\right] e^{\beta z} + \left[\frac{T_0 e^{\beta L} - T_L}{e^{\beta L} - 1}\right]$$

$$T(z) = T_L\left[\frac{e^{\beta z} - 1}{e^{\beta L} - 1}\right] + T_0\left[\frac{e^{\beta L} - e^{\beta z}}{e^{\beta L} - 1}\right]$$

note $$\frac{T(z) - T_0}{T_L - T(z)} = \frac{1 - e^{-\beta z}}{e^{\beta(L-z)} - 1}$$

also $$q_{TOTAL} = -k_b \frac{\partial T}{\partial z} + \rho_w c_w n V_z T$$

$$= -c_1 k_b \beta e^{\beta z} + \rho_w c_w n V_z c_1 e^{\beta z} + \rho_w c_w n V_z C_2$$

$$= \rho_w c_w n V_z C_2$$

$$= \left[\frac{T_0 e^{\beta L} - T_L}{e^{\beta L} - 1}\right] \rho_w c_w n V_z = \left[\frac{T_0 e^{\beta L} - T_L}{e^{\beta L} - 1}\right] k_b \beta$$

consider limit as $\beta \to 0$. use L'Hopital's rule

$$\lim_{\beta \to 0} \frac{e^{\beta z} - 1}{e^{\beta L} - 1} = \lim_{\beta \to 0} \frac{z e^{\beta z}}{L e^{\beta L}} = \frac{z}{L}$$

$$\lim_{\beta \to 0} \frac{e^{\beta L} - e^{\beta z}}{e^{\beta L} - 1} = \lim_{\beta \to 0} \frac{L e^{\beta L} - z e^{\beta z}}{L e^{\beta L}} = 1 - \frac{z}{L}$$

so $$\lim_{\beta \to 0} T(z) = T_L\left(\frac{z}{L}\right) + T_0\left(1 - \frac{z}{L}\right)$$ which is linear in z and satisfies boundary conditions case when $\beta$ small. let $\beta z < \beta L < 1$ (ie 8% error in $e^{\beta L}$)

then $e^{\beta z} \approx 1 + \beta z + \frac{\beta^2 z^2}{2}$ $e^{\beta L} \approx 1 + \beta + \frac{\beta^2 z^2}{2}$

$$T(z) \approx T_L \left[ \frac{\beta z + \frac{1}{2}\beta^2 z^2}{\beta L + \frac{1}{2}\beta^2 L^2} \right] + T_0 \left[ \frac{\beta(L-z) + \frac{1}{2}\beta^2(L^2-z^2)}{\beta L + \frac{1}{2}\beta^2 L^2} \right]$$

$$\approx T_L \left[ \frac{z + \frac{1}{2}\beta z^2}{L + \frac{1}{2}\beta L^2} \right] + T_0 \left[ \frac{(L-z) + \frac{1}{2}\beta(L^2-z^2)}{L + \frac{1}{2}\beta L^2} \right]$$

$$\approx \frac{(T_L - T_0)(z + \frac{1}{2}\beta z^2)}{(L + \frac{1}{2}\beta L^2)} + T_0$$

$$L T(z) + \tfrac{1}{2}\beta L^2 T(z) \approx T_L z + \tfrac{1}{2}\beta z^2 T_L + T_0(L-z) + \tfrac{1}{2}\beta(L^2-z^2)T_0$$

$$\beta \left[ \tfrac{1}{2}L^2 T(z) - \tfrac{1}{2}z^2 T_L - \tfrac{1}{2}(L^2-z^2)T_0 \right] \approx T_L z + T_0(L-z) - L T(z)$$

$$\beta \approx \frac{T_L z + T_0(L-z) - L T(z)}{\frac{1}{2}L^2 T(z) - \frac{1}{2}z^2 T_L - \frac{1}{2}(L^2-z^2)T_0}$$ good when $\beta L < 1$

note ratio of conducted to advected heat is

$$\frac{q_{\text{conducted}}}{q_{\text{advected}}} = \frac{k_b |\nabla T|}{\rho_w c_w \eta T |V|}$$

suppose we pick an average $T = \frac{1}{2}(T_L - T_0)$ and a average gradien $|\nabla T| = \frac{(T_L - T_0)}{L}$

then if $\frac{q_{\text{convected}}}{q_{\text{advected}}} = 1 \approx \frac{1}{\beta} \frac{(T_L - T_0)}{L} \frac{2}{(T_L - T_0)}$

we have $\beta L \approx 2$. if $\beta L < 1$ then $\frac{q_{\text{advected}}}{q_{\text{conducted}}} < 2$

total heat flow when $\beta$ is small

$$q_{TOTAL} = \left[ \frac{T_0 e^{\beta L} - T_L}{e^{\beta L} - 1} \right] \beta K_b$$

$$= \left[ \frac{T_0 \left(1 + \beta L + \frac{1}{2}\beta^2 L^2\right) - T_L}{\left(1 + \beta L + \frac{1}{2}\beta^2 L^2\right) - 1} \right] \beta k_b$$

$$= \left[ \frac{(T_0 - T_L) + T_0 \beta L + \frac{1}{2} T_0 \beta^2 L^2}{\beta L + \frac{1}{2}\beta^2 L^2} \right] \beta K_b$$

$$= + k_b \left[ \frac{(T_0 - T_L) + T_0 L \beta + \frac{1}{2} T_0 \beta^2 L^2}{L + \frac{1}{2}\beta L^2} \right]$$

note

$$\lim_{\beta \to 0} q_{total} = - k_b \frac{(T_L - T_0)}{L}$$

which is correct result for the linear gradient units

$$\beta = \frac{n \rho_w C_w V_z}{k_b}$$

$K_b = \frac{cal}{cm\ sec\ {}^oK}$ $\rho_w = \frac{gm}{cm^3}$ $C_w = \frac{cal}{gm\ {}^oK}$ $V_z = \frac{cm}{sec}$ $n$ = dimensionless

$$\beta = \frac{\cancel{gm}}{cm^3} \cdot \frac{\cancel{cal}}{\cancel{gm}\ \cancel{{}^oK}} \cdot \frac{\cancel{cm}}{\cancel{sec}} \cdot \frac{\cancel{sec}\ \cancel{cm}\ \cancel{{}^oK}}{\cancel{cal}} = \frac{1}{cm}$$

$$V_z = \frac{k_b}{n \rho_w C_w} \beta$$

$\rho_w = 1.04\ gm/cm^3$

$C_w = 0.9\ cal/gm\ {}^oK$

$n = 1.0$ (ie surface velocity)

$k_b = 0.672\ w/m{}^oK$ $10^{-2} \frac{m}{cm}$

$= 6.72 \times 10^{-3}\ w/cm{}^oK$ $\frac{w}{cal/sec} = 0.2389$

$= 1.6054 \times 10^{-3}\ cal/sec\ cm\ {}^oK$

(data for core 144)

$$\frac{1.6054 \times 10^{-3}}{(0.9)(1.04)(1.0)} = 1.76 \times 10^{-3}\ \beta$$

$=$ $V_{min}$ $1.55 \times 10^{-6}$
$V_{BEST} =$ $1.76 \times 10^{-6}$ cm/sec
$V_{max}$ $1.90 \times 10^{-6}$