TRANSFORMATION of variables to cluster grid spacing near origin

MRN 12B

$$r = s R^{\ell} \qquad R = r^{1/\ell} s^{-1/\ell}$$

$$\frac{\partial}{\partial r} = \frac{\partial R}{\partial r}\frac{\partial}{\partial R} = \frac{s^{-1/\ell}}{\ell} r^{\frac{1}{\ell}} r^{-1} \frac{\partial}{\partial R} = \frac{1}{\ell s} R R^{-\ell} \frac{\partial}{\partial r}$$

$$\frac{\partial}{\partial r} = \frac{1}{\ell s} R^{-\ell+1} \frac{\partial}{\partial R}$$

$$\frac{\partial^2}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right) = \frac{1}{\ell s} R^{-\ell+1} \frac{\partial}{\partial R}\left[\frac{1}{\ell s} R^{-\ell+1} \frac{\partial}{\partial R}\right]$$

$$= \frac{1}{s^2 \ell^2} R^{-\ell+1}\left[R^{-\ell+1}\frac{\partial^2}{\partial R^2} + (-\ell+1) R^{-\ell} \frac{\partial}{\partial R}\right]$$

$$= \frac{1}{s^2 \ell^2} R^{-2(\ell-1)} \frac{\partial^2}{\partial R^2} - \frac{(\ell-1)}{s^2 \ell^2} R^{-2\ell+1} \frac{\partial}{\partial R}$$

<u>example</u> $\ell = 2$, $s = 1$

$$\frac{\partial}{\partial r} = \frac{1}{2} R^{-1} \frac{\partial}{\partial R}$$

$$\frac{\partial^2}{\partial r^2} = \frac{1}{4} R^{-2} \frac{\partial^2}{\partial r^2} - \frac{1}{4} R^{-3} \frac{\partial}{\partial R}$$

<u>Transformation of equation</u> $\beta \frac{\partial P}{\partial t} = K\left[\frac{1}{r}\frac{\partial P}{\partial r} + \frac{\partial^2 P}{\partial r^2}\right] + \frac{\partial K}{\partial r} \cdot \frac{\partial P}{\partial r}$

$$r^{-1}\frac{\partial}{\partial r} P = R^{-\ell} \frac{1}{\ell} R^{-\ell+1} \frac{\partial}{\partial R} P$$

$$= \frac{1}{s\ell} R^{-2\ell+1} \frac{\partial P}{\partial R}$$

$$\frac{\partial^2 P}{\partial r^2} = \frac{1}{s^2 \ell^2} R^{-2(\ell-1)} \frac{\partial^2 P}{\partial r^2} - \frac{(\ell-1)}{\ell^2} R^{-2\ell+1} \frac{\partial P}{\partial r}$$

$$r^{-1}\frac{\partial P}{\partial r} + \frac{\partial^2 P}{\partial r^2} = \frac{1}{s^2 \ell^2} R^{-2(\ell-1)} \frac{\partial^2 P}{\partial r^2} + \frac{1}{s^2 \ell^2} R^{-2\ell+1} \frac{\partial P}{\partial r}$$

$$\frac{\partial K}{\partial r}\frac{\partial P}{\partial r} = \frac{1}{s^2 \ell^2} R^{-2(\ell-1)} \frac{\partial K}{\partial R}\frac{\partial P}{\partial R}$$

Transformed equation:

$$s^2 l^2 \beta \frac{\partial P}{\partial t} = R^{-2(l-1)} \frac{\partial^2 P}{\partial r^2} + R^{-2l+1} \frac{\partial P}{\partial R} + R^{-2(l-1)} \frac{\partial K}{\partial R} \frac{\partial P}{\partial R}$$

boundary conditions $\frac{\partial P}{\partial r}(r = r_0) = 0$ $\frac{\partial P}{\partial R}(R = R_0 = r_0^{1/l} s^{-1/l} = 0)$

$P(r = r_{max}) = 0$ $P(R = R_{max} = r_{max}^{1/l} s^{-1/l}) = 0$

<u>finite differences:</u>

$$\frac{\partial P}{\partial R} = \frac{1}{4 \Delta R} \left(P_{m+1}^{n+1} - P_{m-1}^{n+1} + P_{m+1}^{n} - P_{m-1}^{n} \right)$$

$$\frac{\partial^2 P}{\partial R^2} = \frac{1}{2 \Delta R^2} \left(P_{m+1}^{n+1} - 2P_m^{n+1} + P_{m-1}^{n+1} + P_{m+1}^{n} - 2P_m^{n} + P_{m-1}^{n} \right)$$

$$\frac{\partial K}{\partial R} = \frac{1}{2 \Delta R} \left(K_{m+1} - K_{m-1} \right)$$

$$\frac{\partial P}{\partial t} = \frac{1}{\Delta t} \left(P_m^{n+1} - P_m^{n} \right)$$

<u>equation becomes:</u>

$$0 = -s^2 l^2 \beta \frac{\partial P}{\partial t} + R^{-2(l-1)} \frac{\partial^2 P}{\partial R^2} + \left(R^{-2l+1} + R^{-2(l-1)} \frac{\partial K}{\partial R} \right) \cdot \frac{\partial P}{\partial R}$$

$$0 = a_m \frac{\partial P}{\partial t} + b_m \frac{\partial^2 P}{\partial R^2} + c_m \frac{\partial P}{\partial R}$$

$a_m = -s^2 l^2 \beta$ $b_m = R^{-2(l-1)}$

$$c_m = \left[R^{-2l+1} + R^{-2(l-1)} \left(K_{m+1} - K_{m-1} \right) / 2 \Delta R \right]$$

$$P_{m+1}^{n+1}\left[\frac{b_m}{2\Delta R^2}+\frac{c_m}{4\Delta R}\right]$$

$$+\ P_m^{n+1}\left[\frac{a_m}{\Delta t}-\frac{b_m}{\Delta R^2}\right]$$

$$+\ P_{m-1}^{n+1}\left[\frac{b_m}{2\Delta R^2}-\frac{c_m}{4\Delta R}\right] =$$

$$-\ P_{m+1}^{n}\left[\frac{b_m}{2\Delta R^2}+\frac{c_m}{4\Delta R}\right]$$

$$-\ P_m^{n}\left[\frac{-a_m}{\Delta t}-\frac{b_m}{\Delta R^2}\right]$$

$$-\ P_{m-1}^{n}\left[\frac{b_m}{2\Delta R^2}-\frac{c_m}{4\Delta R}\right]$$

Tridiagonal equation: $A_m P_{m-1}^{n+1} + B_m P_m^{n+1} + C_m P_{m+1}^{n+1} = F[P^n]$

$$C_m = \left[\frac{b_m}{2\Delta R^2}+\frac{c_m}{4\Delta R}\right]$$

$$B_m = \left[\frac{a_m}{\Delta t}-\frac{b_m}{\Delta R^2}\right] \qquad D_m = \left[\frac{a_m}{\Delta t}+\frac{b_m}{\Delta R^2}\right]$$

$$A_m = \left[\frac{b_m}{2\Delta R^2}-\frac{c_m}{4\Delta R}\right]$$

$$F[P^n] = -A_m P_{m-1}^n + D_m P_m^n - C_m P_{m-1}^n .$$