Computing Model Variance Without Computing the Complete Generalized Inverse

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1. For uncorrelted data with uniform variance σ_d^2, the covariance of the estimated model parameters is:

$$\text{cov}\,\mathbf{m}^{\text{est}} = \sigma_d^2 \mathbf{G}^{-g}\mathbf{G}^{-gT}$$

2. Note that the variance of the k-th model parameter is

$$\text{var}\, m_k = \sigma_d^2 \sum_i G_{ki}^{-g} G_{ik}^{-gT} = \sigma_d^2 \sum_i G_{ki}^{-g} G_{ki}^{-g}$$

Thus, the variance equals σ_d^2 multiplied by the k-th row of the generalized inverse, dotted into itself.

3. In the case of the Backus-Gilbert generalized inverse, each row of $\mathbf{G}^{-g}$ is computed separately. So after we compute its k-th row, we can calculate var m_k trivially.

4. The damped least squares generalized inverse has the form

$$\mathbf{G}^{-g} = [\mathbf{G}^{\mathbf{T}}\mathbf{G} + \varepsilon^2\mathbf{H}^{\mathbf{T}}\mathbf{H}]^{-\mathbf{1}}\mathbf{G}^{\mathbf{T}} = \mathbf{A}^{-\mathbf{1}}\mathbf{G}^{\mathbf{T}} \quad \text{with} \quad \mathbf{A} = \mathbf{G}^{\mathbf{T}}\mathbf{G} + \varepsilon^2\mathbf{H}^{\mathbf{T}}\mathbf{H}$$

Note that since $\mathbf{A}$ is symmetric, its inverse will also be symmetric. The relevant issue is how to calculate the k-th row of the generalized inverse $\mathbf{G}^{-g}$ without having to compute the others.

5. Note that the equation

$$\mathbf{A}\,\mathbf{A}^{-\mathbf{1}} = \mathbf{I} \quad \text{or} \quad \sum_k A_{ik} A_{kj}^{-1} = \delta_{ij}$$

can be read as a sequence of vector equations

$$\mathbf{A}\,\mathbf{v}^{(\mathrm{j})} = \mathbf{s}^{(\mathrm{j})} \quad \text{with} \quad \left[\mathbf{v}^{(\mathrm{j})}\right]_{\mathbf{i}} = \left[\mathbf{A}^{-\mathbf{1}}\right]_{\mathbf{ij}} \quad \text{and} \quad \left[\mathbf{s}^{(\mathrm{j})}\right]_{\mathbf{i}} = \delta_{ij}$$

That is, $\mathbf{v}^{(\mathrm{j})}$ is the j-th column of $\mathbf{A}^{-\mathbf{1}}$ and $\mathbf{s}^{(\mathrm{j})}$ is the corresponding column of the identity matrix. Hence we can solve for the j-th column of $\mathbf{A}^{-\mathbf{1}}$ by solving the system $\mathbf{A}\,\mathbf{v}^{(\mathrm{j})} = \mathbf{s}^{(\mathrm{j})}$. Furthermore, since $\mathbf{A}^{-\mathbf{1}}$ is symmetric, we $\mathbf{v}^{(\mathrm{j})\mathrm{T}}$ its j-th row.

6. Now notice that

$$\mathbf{G}^{-g} = \mathbf{A}^{-\mathbf{1}}\,\mathbf{G}^{\mathrm{T}} \quad \text{or} \quad [\mathbf{G}^{-g}]_{kj} = \sum_i A_{ki}^{-1} G_{ij}^{T}$$

Hence, the k-th row of the generalized inverse $\mathbf{G}^{-\mathrm{g}}$ it's the k-th row of $\mathbf{A}^{-\mathbf{1}}$ dotted into $\mathbf{G}^{\mathrm{T}}$.

Example (with MATLAB code appended below)

Error in inverse AI: 0.000000

Error in geneneralize inverse GMG: 0.000000

Error in variance vm: 0.000000

```
% tests procedure for calculating the variance of a single
% estimated model parameter.

clear all;

% under-determined problem
N=10;
M=2*N;

% hypothetical data kernel
s=0.1;
G=random('Normal',0, s, N, M ) + eye(N,M);

% stddev of data
sigmad = 1;

% use damped least squares generalized inverse
% GMG = (G'*G + epsi*eye(M,M)) \ G'
epsi = 0.1;
A = (G'*G + epsi*eye(M,M));

% calculate the full version of various matrices,
% which are only needed to check the results of the calculation
% AI: inverse of A
% GMG: generalized inverse
% vm: var m
AI = inv(A);
GMG = A \ (G');
Cm = (sigmad^2) * GMG*GMG';
vm = diag(Cm);

% the target model parameter whose varoiance we wish
% to calculate separately

k=M/2;

% use k-th column of identity matrix to solve for
% k-th column of AI (or row, since its symmetric)
s = zeros(M,1);
s(k) = 1.0;
aik = A \ s;

% check that the calculation was correct
eai = aik-AI(:,k);
Eai = eai'*eai;
fprintf('Error in inverse AI: %f\n', Eai );

% calculate the k-th row of the generalised inverse
gmgk = (aik') * (G');

% check that the calculation was correct
egmg = (gmgk - GMG(k,:))';
Egmg = egmg'*egmg;
fprintf('Error in geneneralize inverse GMG: %f\n', Egmg );
```

```
% calculate the k-th variance
vmk = (sigmad^2) * gmgk*gmgk';

% check that the calculation was correct
evm = vmk-vm(k);
Evm = evm^2;
fprintf('Error in variance vm: %f\n', Evm );
```