Some Notes Relevant to Anisotropic Tomography.

Bill Menke, April 1-15, 2014

① 2D Fourier Transform applied to spike isotropic heterogeneity using projection slice Thm. and mapped to anisotropic heterogeneity. Only able to invert one integral, but note r^{-2} dependence

② Two integrals relevant to ① from G. & R.'s Integral Book

③ Two more integrals ...

④,⑤ proof That a certain line integral is zero

⑥ normalization of ray integral

⑦ proof another ray integral is zero.

⑧ integral, incomplete, not used.

Fourier Transform, polar coords

$$\hat{g}(f_x, f_y) = \iint_{-\infty}^{+\infty} g(x,y) \exp\{-2\pi i(f_x x + f_y y)\} dx\, dy$$

$$\hat{g}(k_x, k_y) = \iint_{-\infty}^{+\infty} g(x,y) \exp\{-i(k_x x + k_y y)\} dx\, dy$$

$$k_x = 2\pi f_x \quad k_y = 2\pi f_y$$

Inverse FT

$$\hat{g}(\rho, \phi) = \int_0^\infty r\, dr \int_0^{2\pi} d\theta \; g(r,\theta)\, e^{+2\pi i \rho r \cos(\phi-\theta)}$$

$$g(r,\theta) = \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\phi \; \hat{g}(\rho,\phi)\, e^{-2\pi i \rho r \cos(\phi-\theta)}$$

$k_r = 2\pi\rho$

$$\hat{g}(k_r, \phi) = \int_0^\infty r\, dr \int_0^{2\pi} d\theta \; g(r,\theta)\, e^{+i k_r r \cos(\phi-\theta)}$$

$$g(r,\theta) = \frac{1}{(2\pi)^2} \int_0^\infty k_r\, dk_r \int_0^{2\pi} d\phi \; \hat{g}(k_r,\phi)\, e^{-i k_r r \cos(\phi-\theta)}$$

proposed F.T. let $\hat{g}(k_r, \phi) = \cos(2\phi) = \cos^2\phi - \sin^2\phi$

$$g(r,\theta) = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\phi \cos\phi \int_0^\infty k_r \begin{smallmatrix}\cos \\ i\sin\end{smallmatrix} \{k_r \overbrace{r\cos(\phi-\theta)}^{a}\}\, dk_r$$

Attemp at inversion

form $\int_0^\infty x \cos(ax)\, dx = -\frac{1}{a^2}$

$i\int_0^\infty x \sin ax\, dx = 0$

$$\frac{1}{(2\pi)^2} \int_0^{2\pi} \cos 2\phi \frac{1}{r^2 \cos^2(\phi-\theta)}\, d\phi$$

$$\frac{r^{-2}\cos 2\phi}{\cos\phi\cos\theta - \sin\phi\sin\theta}$$

$$= \frac{r^{-2}\cos 2\phi}{\cos(\phi-\theta)}$$

$$= r^{-2}\cos 2\theta \frac{\cos 2y}{\cos y} - r^{-2}\sin 2\theta \frac{\sin 2y}{\cos y}$$

$y = \phi - \theta$

$\phi = y + \theta$

$\cos 2\phi = \cos(2y + 2\theta)$

$\cos 2\theta \cos 2y - \sin 2\theta \sin 2y$

3.944.5

$$I = \int_0^\infty x^{\mu-1} e^{-\beta x} \sin \delta x \, dy = \frac{\Gamma(\mu)}{(\beta^2+\delta^2)^{\frac{\mu}{2}}} \sin\left(\mu \tan^{-1} \frac{\delta}{\beta}\right)$$

$\mathrm{re}\, \mu > -1 \quad \mathrm{Re}\, \beta > \mathrm{Im}\, \delta$

$\mu = 2$ so $\mu - 1 = 1 \quad \frac{\mu}{2} = 1$

$\Gamma(2) = (2-1)! = 1! = 1$

$\beta \to 0$

$$I = \frac{1}{\delta^2} \sin\left(2 \tan^{-1} \infty\right) = \frac{1}{\delta^2} \sin\left(2 \frac{\pi}{2}\right) = 0$$

3.944.6

$$I = \int_0^\infty x^{\mu-1} e^{-\beta x} \cos \delta x \, dx = \frac{\Gamma(\mu)}{(\delta^2+\beta^2)^{\frac{\mu}{2}}} \cos\left(\mu \tan^{-1} \frac{\delta}{\beta}\right)$$

$$= \frac{1}{\delta^2} \cos\left(2 \frac{\pi}{2}\right) = -\frac{1}{\delta^2}$$

2.539.4

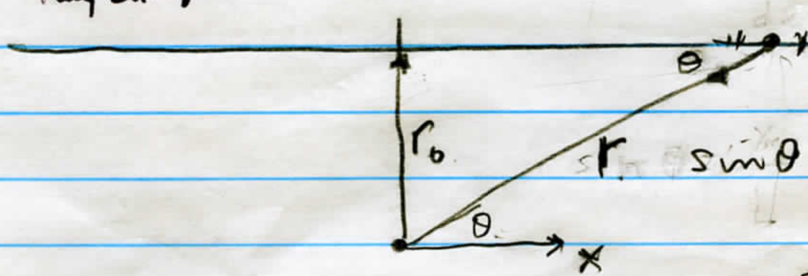
$$\int \frac{\cos 2x}{\cos x}\,dx = 2\sin x - \ln \operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

2.539.1 $\int \frac{\sin 2x\,dx}{\cos^n x} = \frac{-2}{(n-2)\cos^{n-2} x} \xrightarrow[n=1]{} -2\cos x$

$n-2=-1$

$$\frac{\sin 2x}{\cos x} = \frac{2\sin x\cos x}{\cos x} = 2\sin x$$

Ray Integral



$$r^2 = l^2 + r_0^2$$

$$\sin\theta = \frac{r_0}{\sqrt{l^2+r_0^2}} = \frac{r_0}{r}$$

$$\cos\theta = \frac{l}{\sqrt{l^2+r_0^2}}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= \frac{l^2}{l^2+r_0^2} - \frac{r_0^2}{l^2+r_0^2}$$

$$= \frac{l^2-r_0^2}{l^2+r_0^2} = \frac{l^2}{l^2+r_0^2} - \frac{r_0^2}{l^2+r_0^2}$$

$$\int_0^\infty \frac{l^2-r_0^2}{(l^2+r_0^2)} R(r_0^2+l^2)\,dl = 0 \qquad \text{Try } R(r) = r^{-2}$$

$$\int_0^\infty \frac{l^2-r_0^2}{(l^2+r_0^2)^{a+1}}\,dl =$$

$$\int_0^\infty \frac{l^2}{(l^2+r_0^2)^2}\,dl - \int_0^\infty \frac{r_0^2}{(l^2+r_0^2)^2}$$

2.175.4 $\quad R = a + bx + cx^2 \quad \Delta = 4ac - b^2$

$$\int \frac{x^2}{R^2}dx = \frac{ab + (b^2 - 2ac)x}{c\Delta R} + \frac{2a}{\Delta}\int \frac{dx}{R}$$

2.173.1

$$\int \frac{1}{R^2}dx = \frac{b + 2cx}{\Delta r} + \frac{2c}{\Delta}\frac{dx}{R}$$

$$I_A = \int_0^\infty \frac{\ell^2}{(\ell^2 + r_0^2)^2} d\ell$$

$x = \ell$
$a = r_0^2$
$b = 0$
$c = 1$
$\Delta = 4r_0^2$

$$\Rightarrow \int_0^\infty \frac{x^2}{R^2}dx$$

$$\Rightarrow \frac{-2r_0^2 x}{\Delta R} + \frac{2r_0^2}{\Delta}\int \frac{dx}{R}$$

$I_B =$

$$-r_0^2 \int_0^\infty \frac{1}{(\ell^2 + r_0^2)^2} \Rightarrow (\text{same}) \Rightarrow -r_0^2 \int_0^\infty \frac{1}{R^2}dx =$$

$$-r_0^2\left(\frac{2x}{\Delta r} + \frac{2}{\Delta}\int_0^\infty \frac{dx}{R}\right)$$

$$-\frac{4r_0^2 x}{\Delta R} - \frac{2r_0^2}{\Delta}\int_0^\infty \frac{dx}{R}$$

$$I_A + I_B = \frac{-4r_0^2 x}{4r_0^2(r_0^2 + x^2)} = -\frac{x}{r_0^2 + x^2}$$

$$\Big|_{x=0} = 0 \qquad \Big|_{x \to \infty} = 0$$

$$2\int_0^\infty \frac{1}{a^2+x^2}\,dx = \frac{2}{a}\tan^{-1}\frac{x}{a}\Big|_0^\infty = \frac{2}{a}\tan^{-1}\infty - 0 = \frac{2}{a}\frac{\pi}{2} = \frac{\pi}{a}$$

$s = \delta(x)\,T_0$

$T = \int s\,dx = T_0$

$$s = \frac{a}{\pi}T_0\left(\frac{1}{a^2+x^2}\right)$$

$$T = \int s\,dx = \frac{a}{\pi}T_0 \; \underbrace{2\int\frac{dx}{a^2+x^2}} = T_0$$

x

$\dfrac{x^4}{x^4+\varepsilon^4}$ $x \ll \varepsilon$ $\dfrac{x^4}{\varepsilon^4}$

$x \gg \varepsilon$ 1

$\dfrac{x^2}{x^4+\varepsilon^4}$ $x \ll \varepsilon$ x^2/ε^4

$x \gg \varepsilon$ x^{-2}

$$2.\int_0^\infty \frac{x^2}{x^4+\varepsilon^4}$$

$$z_4 = \overset{\varepsilon^4}{a} + \overset{1}{b}\,x^4$$

case $ab > 0$

$\alpha = \sqrt[4]{\frac{\varepsilon^4}{1}} = \varepsilon$

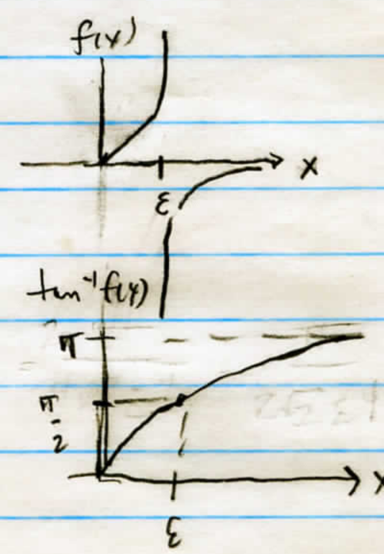
2.132 . 3 $\displaystyle\int\frac{x^2}{z_4}\,dx = \frac{1}{4\varepsilon^4\sqrt{2}}\Big\{$

$$2\cdot\frac{1}{4\varepsilon\sqrt{2}}\left\{\ln\frac{x^2-\varepsilon x\sqrt{2}+\varepsilon^2}{x^2+\varepsilon x\sqrt{2}+\varepsilon^2} + 2\tan^{-1}\overbrace{\boxed{\frac{\varepsilon x\sqrt{2}}{\varepsilon^2-x^2}}}^{f(x)}\right\}$$

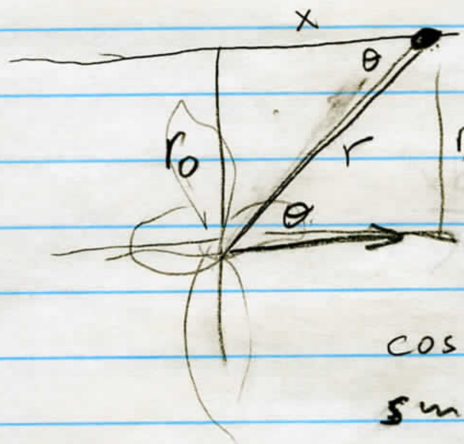
$$x\to 0 \quad 2\cdot\frac{1}{4\varepsilon\sqrt{2}}\left\{\ln 1 + 2\tan^{-1}0\right\} = 0$$

$$x\to\infty \quad 2\cdot\frac{1}{4\varepsilon\sqrt{2}}\left\{\ln 1 + 2\pi\right\} = \frac{\pi}{\varepsilon\sqrt{2}}$$

$$\frac{\varepsilon\sqrt{2}\,T_0}{\pi}\left(\frac{r^2}{r^4+\varepsilon^4}\right)$$



$\cos(a+b)$



$\cos 2(\theta-\theta_0)$

$\cos 2\theta_0 \cos 2\theta + \sin 2\theta_0 \sin 2\theta$

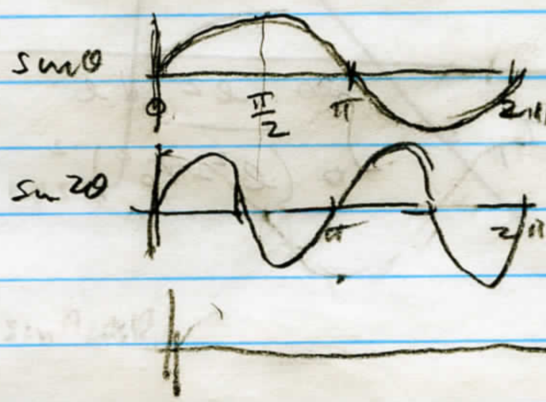
$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2 \sin\theta \cos\theta$

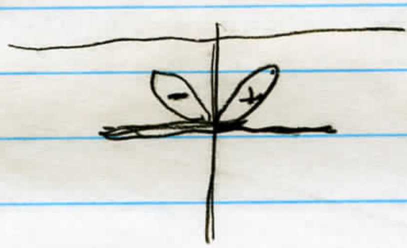
$\sin\theta = \dfrac{r_0}{(r_0+x^2)^{1/2}}$

$\sin\theta\cos\theta = \dfrac{r_0 x}{(r_0+x^2)}$

$\cos\theta = \dfrac{x}{(r_0+x^2)^{1/2}}$



$$\int_{-\infty}^{+\infty} \frac{r_0 x}{(r_0+x^2)^3}\,dx$$



2.557.3

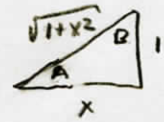
$$\int \frac{\cos x \, dx}{a\cos x + b\sin x} = \int \frac{dx}{a + b\tan x} =$$

$$\frac{ax + b \ln \sin\left(x + \tan^{-1}\frac{a}{b}\right)}{a^2+b^2}$$

$a = \cos\theta \quad b = -\sin\theta \quad a^2+b^2=1 \quad x = \theta \quad \frac{a}{b} = \frac{\cos\theta}{-\sin\theta} = -ctg(\theta)$

$(\cos\theta)\phi + b \ln\sin(\phi + \tan^{-1}(-ctg(\theta)))$

$(\cos\theta)\phi + b \ln\sin(\phi - \frac{\pi}{2} + \theta)$



$\sin A = \frac{1}{r}$

$\cos A = \frac{x}{r}$

$\cot A = x$

$\tan B = x$

$\cot A = \tan B = \tan(\frac{\pi}{2} - A)$

$\tan^{-1}\cot A = \frac{\pi}{2} - A$