Data Smoothing with Exponential C_h Menke May 8 2014 0

consider $(C_h)_{ij} = \varepsilon^{-2} \left[\frac{a}{2}\right] \exp(-a|i-j|)$

and the data smoothing problem $\begin{pmatrix} I \\ C_h^{-1/2} I \end{pmatrix} m = \begin{pmatrix} d \\ 0 \end{pmatrix}$

1. The operator that is equivalent to C_h^{-1} is $\varepsilon^{2}\left(1 - a^{-2}\frac{d^2}{dx^2}\right)$

2. The operator that is equivalent to $C_h^{-1/2}$ is $\varepsilon\left(1 - a^{-1}\frac{d}{dx}\right)$
(but note it is not self-adjoint)

3. The self-adjoint operator that is equivalent to
$C_h^{-1/2}$ is $i\varepsilon\mathcal{H}\left(1 - a^{-1}\frac{d}{dx}\right) = -i\varepsilon a^{-1}\frac{d}{dx}\mathcal{H}$ where $\mathcal{H}$ is Hilbert Transform

4. In the continuum limit
$m(x) = \left(c \exp(-|b|x)\right)$ convolved with $d(x)$
with $c = (1+\varepsilon^2)^{-1}\left(\frac{b}{2}\right)$ and $b = \frac{(1+\varepsilon^2)^{1/2}}{\varepsilon} a$

5. Note autocorrelation (m) not exponential fcn

6. In order for autocorrelation (m) to be exponential fcn, $m = g * d$ with $g \propto K_0(a|x|)$

7. But I haven't worked out $C_h^{-1/2}$ for this case

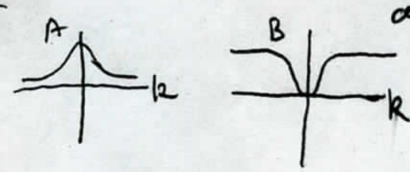
Menke
May 7, 2014
(1)

$A(x) = \frac{a}{2} e^{-a|x|} = \frac{a}{2} f(x)$ $\quad \int_{-\infty}^{+\infty} e^{-a|x|}\, dx = 2\int_0^{\infty} e^{-ax}\, dx = -\frac{2}{a} e^{-ax}\Big|_0^{\infty} = \frac{2}{a}$

$A(k) = 2\frac{a}{2}\int_0^{\infty} e^{-a|x|} \cos kx\, dx = \frac{a^2}{k^2+a^2}$ see GR 3.893.2

note $\lim_{k\to 0} A(k) = 1$ ie unit area

$\frac{1}{A(k)} = \frac{k^2+a^2}{a^2} = B(k)$



$$B(x) = \frac{1}{(2\pi)}\frac{1}{(a^2)}\int_{-\infty}^{\infty} (k^2+a^2)\cos kx\, dk$$

$$= \frac{1}{a^2}\left(-\ddot{\delta}(x) + a^2\delta(x)\right)$$

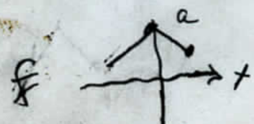
Ift $1 \to \delta(x)$
Ift $-k^2 \to \ddot{\delta}(x)$

$$B(x)* = \frac{1}{a^2}\left(-\frac{d^2}{dx^2} + a^2\right) = \frac{1}{a^2}\mathcal{L} = 1 - \frac{1}{a^2}\frac{d^2}{dx^2}$$

$x>0$ $\quad f = e^{-ax}$ $\quad \mathcal{L}f = \left(-\frac{d^2 f}{dx^2} + a^2 f\right) = \left(-a^2 e^{-ax} + a^2 e^{-ax}\right) = 0$

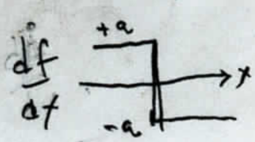
$x<0$ $\quad f = e^{+ax}$ $\quad \mathcal{L}f = \quad$ " $\quad = \left(-a^2 e^{+ax} + a^2 e^{+ax}\right) = 0$

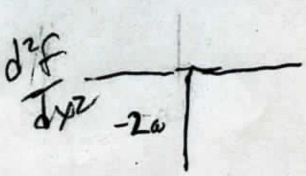
$x=0$



so $\mathcal{L}f = -2a\delta(x)$ and

$A*B = \frac{1}{a^2}\mathcal{L}\frac{a}{2}f =$
$\frac{1}{a^2}\frac{a}{2}\mathcal{L}f = \frac{1}{a^2}\left(\frac{a}{2}\right)(-2a)\delta(x)$
$= \delta(x)$





upshot operator that inverts convolution by

$A(x) = \frac{a}{2}\exp(-a|x|)$

is $1 - \frac{1}{a^2}\frac{d^2}{dx^2}$

$$\left(G^T C_d^{-1} G + H^T C_h^{-1} H\right) m = G^T C_d^{-1} d + 0$$ (2)

$G = I \qquad H = I \qquad \underline{\underline{C_h}} = \varepsilon^{-2} \underline{\underline{A}} \qquad A_{ij} = \frac{a}{2} e^{-a|x_i - x_j|}$

$C_d = I$

$$\left(\varepsilon^2 \underline{\underline{A}}^{-1} + \underline{\underline{I}}\right) \underset{\sim}{m} = \underline{d} \qquad \text{let } \underline{\underline{B}} = \underline{\underline{A}}^{-1}$$

now take continuum limit

from page 1.

$$B = 1 - a^{-2}\frac{d^2}{dx^2}$$

$$\varepsilon^2 B * m + m = d$$

$$\varepsilon^2\left(m - a^{-2}\frac{d^2 m}{dx^2}\right) + m = d$$

$$-\varepsilon^2 a^{-2} \frac{d^2 m}{dx^2} + (1+\varepsilon^2) m = d \qquad d = \delta \text{ to get G.F.}$$

$$\frac{d^2 m}{dx^2} - \underbrace{\frac{a^2(1+\varepsilon^2)}{\varepsilon^2}}_{b^2} m = -\delta(x)\frac{a^2}{\varepsilon^2} \qquad b = \frac{(1+\varepsilon^2)^{1/2}}{\varepsilon} a$$

$$\mathcal{L} m = \frac{d^2 m}{dx^2} - b^2 m = -\frac{a^2}{\varepsilon^2}\delta(x)$$

$$\frac{b^{-1}}{a^{-1}} = \frac{a}{b} = \frac{\varepsilon}{(1+\varepsilon^2)^{1/2}}$$

$\dfrac{\text{output scale length}}{\text{input scale length}}$

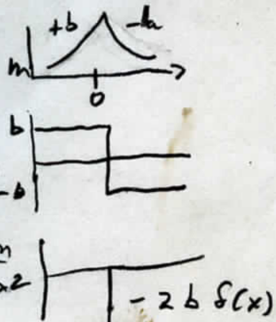
try $m = c e^{-b|x|}$

$x < 0 \quad m = c e^{bx} \quad \mathcal{L} m = c b^2 e^{bx} - c b^2 m = 0$ ✓

$x > 0 \quad m = c e^{-bx} \quad \mathcal{L} m = c b^2 e^{-bx} - c b^2 e^{-bx} = 0$ ✓

$$\frac{a^2}{b^2} = \frac{\varepsilon^2}{(1+\varepsilon^2)}$$

$x = 0$



$\mathcal{L} m = (c)(-2b)\,\delta(x) = -\dfrac{a^2}{\varepsilon^2}\delta(x) \qquad$ so $c = \dfrac{a^2}{\varepsilon^2 2b}$

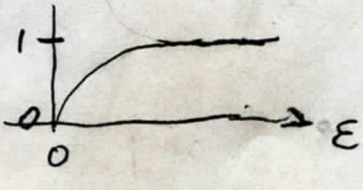
$$= \frac{a^2}{b^2 \varepsilon^2}\frac{b}{2} = (1+\varepsilon^2)^{-1}\frac{b}{2}$$

$$m = (1+\varepsilon^2)^{-1}\frac{b}{2} e^{-b|x|}$$

$\varepsilon \to 0 \quad g \to \delta$

$\varepsilon \to \infty \quad g \to 0$

$$\frac{a}{b} = \frac{b^{-1}}{a^{-1}}$$



if $m = f * d$ then $a_m(x) = f(-t) * d(-t) * f(t) * d(t) = (f \star f) * a_d$

so if $a_d = \delta$ $a_m = f \star f$

a symmetric function has autocorrelation fcn $e^{-\alpha|x|}$

what's the function, $f(x)$?

$(f \star f)(x) = a(x) = f * f(-x)$

fourier transform

since sym fcn has real FT

$a(k) = f^*(k)\, f(k) = f^2(k)$

$$\boxed{\begin{array}{c}\text{upshot} \\ K_0 * K_0 \propto e^{-|x|}\end{array}}$$

$a(k) = 2\int_0^\infty e^{-\alpha|x|} \cos(kx)\, dx$

GR 3.893, 2

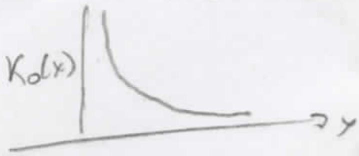
$\int_0^\infty e^{-px} \cos(qx+\lambda) = \frac{1}{p^2+q^2}(p\cos\lambda - q\sin\lambda)$

$a(k) = 2\frac{\alpha}{k^2+\alpha^2}$ $\qquad f(k) = \frac{\sqrt{2\alpha}}{(k^2+\alpha^2)^{1/2}}$

$f(x) = \frac{1}{2\pi}\, 2\int_0^\infty \frac{\sqrt{2\alpha}}{(k^2+\alpha^2)^{1/2}} \cos kx\, dx$

GR 3.754.2

$\int_0^\infty \frac{\cos \alpha x}{(\beta^2+x^2)^{1/2}}\, dx = K_0(\alpha\beta)$



$f(x) = \frac{\sqrt{2\alpha}}{\pi} K_0(\alpha|x|)$

check GR 6.671.14

$\int_0^\infty K_0(\beta x)\cos(ax)\, dx = \frac{\pi}{2\sqrt{\alpha^2+\beta^2}}$

$\int_0^\infty K_0(\alpha k)\cos(kx)\, dx = \frac{\pi}{2\sqrt{x^2+\alpha^2}}$

Wikipedia Screened Poisson Eqn

$$\left(\Delta_\perp - \frac{1}{\rho^2}\right) u(r_\perp) = -f(r_\perp)$$

$$\Delta_\perp = \nabla\cdot\nabla_\perp \quad \nabla_\perp = \nabla - \frac{\underline{B}}{B}\cdot\nabla = \nabla - \hat{z}\cdot\nabla$$

$$G(r_\perp) = \frac{1}{2\pi} K_0\left(\frac{r_\perp}{\rho}\right) \quad \text{for } \left(k_\perp^2 + \frac{1}{\rho^2}\right) G(k_\perp) = 1 \quad \text{implies } -\frac{\delta(r)}{2\pi r}$$

$$\Delta_\perp = \nabla\cdot\nabla - \nabla\cdot(\hat{t}\cdot\nabla) \longrightarrow 2D\ \nabla^2 \longrightarrow \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial}{\partial r}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$$

$$\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - \frac{1}{\rho^2}\right) m = -\frac{\delta(r)}{2\pi r}$$

$$\underbrace{\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - \left(\frac{1}{\rho^2}+1\right)\right\}}_{C_h^{-1}} m + m = -\frac{\delta(r)}{2\pi r}$$

$$\Delta_\perp u - \frac{1}{\rho^2} u$$

$$\frac{d^2}{dr^2} u + \frac{1}{r}\frac{d}{dr} u - \frac{1}{\rho^2} u$$

$\times r^2$

$$r^2\frac{d^2}{dr^2} u + r\frac{d}{dr} u - \frac{r^2}{\rho^2} u$$

wikipedia: Bessel Function:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+\alpha^2) y = 0$$

$$H^T C_h^{-1} H = H^T A^T A^{\,} H = (A H)^T (A H)$$

so one must factorize C_h^{-1} as $A^T A$

Switching to operator, some must factorize

$\mathcal{C}_h^{-1} = 1 - a^{-2} \frac{d}{dx^2}$ so Tha $\mathcal{C}_h^{-1} = \mathcal{A}^+ \mathcal{A}$.

try $\mathcal{A} = (1 + a^{-1} \frac{d}{dx})$ $\mathcal{A}^+ = (1 - a^{-1} \frac{d}{dx})$

$$\mathcal{A}^+ \mathcal{A} = (1 - a^{-1} \tfrac{d}{dx})(1 + a^{-1} \tfrac{d}{dx}) = 1 - a^{-2} \tfrac{d^2}{dx^2}) \checkmark$$

but $\mathcal{A}$ is not self adjoint, However, for matrices, we could symmetrize A by a unitary transformation $\mathcal{U}$ such That $\mathcal{U}^T \mathcal{U} = I$, $C_m^{-1} = A^T I A = A^T \mathcal{U}^T \mathcal{U} A = (\mathcal{U}A)^T (\mathcal{U}A)$. So we need a linear operator that is unary in The same sense. Try i times The Hilbert transform $i\mathcal{H}$, sine $i\mathcal{H}(i\mathcal{H} m) = (i)^2 (-m) = m$

$$i\mathcal{H}(1 - c^{-1} \tfrac{d}{dx}) = -i a^{-1} \mathcal{H}_1 (\tfrac{d}{dx}) \quad (\text{since } \mathcal{H}(\text{const}) = 0)$$

$$\mathcal{H}(u) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{u(t)}{x-t} dt \quad \text{(Wikipedia "Hilbert Transform")}$$

$$\mathcal{H}\left(\frac{du}{dx}\right) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \underbrace{(x-t)^{-1}}_{v} \underbrace{\frac{du(t)}{dt} dt}_{du} = \frac{1}{\pi} \underbrace{(x-t)^{-1}}_{v} \underbrace{u}_{u} \Big|_{-\infty}^{+\infty} - \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{u(t)}_{u} \underbrace{(x-t)^{-2}; dt}_{dv}$$

$\hookrightarrow = \frac{dv}{db} dt$

But dubious whether convolution by $-\frac{1}{\pi} \frac{1}{x^2}$ is well-defined. It seems to diverge.

note $\mathcal{H}(\frac{du}{dx}) = \frac{d}{dx} \mathcal{H}(u)$. The finite difference derivative of the DHT $\begin{cases} 0 & \text{even } n \\ \frac{2}{\pi n} & \text{odd } n \end{cases}$ compares well with matrix C^{-1} except for odd-even chatter.

The matrix $B = I - a^{-2} D_2$ where D_2 = second difference does well in satisfying $C = B^{-1}$ and The matrix $A = I - a^{-1} D_1$ does well in satisfying $C = (A^T A)^{-1}$, for 100×100 matrices, except for edge effects.