Proof of Concept: P – S Velocity Ratio from Seismic Image Via Sparsity Inversion

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after a thought-provoking seminar by Renata Wentzcovitch

General idea: One has images of seismic velocity perturbations, Δv_p and Δv_s. One hypothesizes that they contain geographical regions where Δv_p and Δv_s scale with some fixed but unknown ratio, and other patches where they do not scale. One wants to identify the patches where they scale and determine the ratio.

For simplicity, I implement this proof-of-concept example as a one-dimensional problem in a position variable, z_i $(1 \le i \le N)$, and not as a two (or three) dimensional image. Call the observed velocity perturbations, $d_i^{(A)}$ and $d_i^{(B)}$, where d is for "data". Suppose that $N = N_1 \times N_2$, where there are N_1 geographical patches each of size N_2. The model that I consider is

$$d_i^{(B)} = m_i^{(A)} + m_j^{(B)} d_i^{(A)} \quad \text{where} \quad j = \mathrm{mod}(i, N_2) + 1 \tag{1}$$

Here, m is for "model parameter" (the unknowns). The total number of model parameters is $M = N + N_1$. There are N model parameters, $m_i^{(A)}$, one for every position, and there are N_1 model parameters, $m_i^{(B)}$, one for every geographical patch. The $d_i^{(B)}$ data can be constructed in two ways: in a geographical patch where they *do not* scale with $d_i^{(A)}$, from the $m_i^{(A)}$, which specify their value; and in a geographical patch where they *do* scale with $d_i^{(A)}$, from the $m_j^{(B)}$, which specifies the ratio of $d_i^{(A)}$ to $d_i^{(B)}$. The inverse problem is under-determined, with N_1 more unknowns than data.

My proposal is to resolve the under-determinacy by adding prior information of sparsity; that is, to make as many model parameters zero as possible. In patches where the data scale, sparsity implies that the $m_i^{(A)}$ are zero. In patches where the data do not scale, sparsity implies that the $m_i^{(B)}$ are zero. I use a re-weighting scheme that solves the problem

$$\text{find the } \mathbf{m} \text{ that minimizes } \left\|\mathbf{d}^{(B)} - \mathbf{G}\mathbf{m}\right\|_2^2 + \|\mathbf{m}\|_0^0 \tag{2}$$

where $\mathbf{m} = [\mathbf{m}^{(A)} \quad \mathbf{m}^{(B)}]^T$ and $\mathbf{G}$ is a matrix that implements Eq. (1). Note that $\mathbf{G}$ depends upon $\mathbf{d}^{(A)}$. In practice, the L_0 norm, $\|.\|_0^0$, is approximated with the $L_{0.1}$ norm.

A numerical experiment is shown in Fig. 1 and Fig. 2. The various weighting parameters in the inversion need to be tuned manually and with some care, but the estimated solution is close to the true one.

A limitation of this approach is the geographical regions need to be specified; they are not chosen by the inversion.

Although I have modeled the ratio as being piecewise-constant in the geographical regions, one could imagine using a more complicated spline representation (as long as the spline can be exactly zero in a region).

I have used data that are the velocity perturbations, themselves, which leads to a rather simple **G**. However, the method is completely general. One could easily substitute a **G** that linked any kind of data to the model parameters (such as travel time data). Thus, the method is not restricted to post-processing images that arise, say, from seismic tomography. It could be built into the tomography, itself.

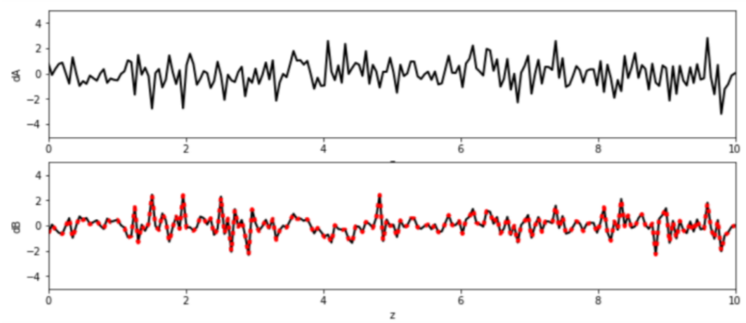

Fig. 1 Synthetic data, $d_i^{(A)}$ and $d_i^{(B)}$, plotted as a function of position, z_i. In this test, $N_1 = 10$, $N_2 = 20$ and the geographical regions are 1 z-unit wide. (Top) The true $d_i^{(A)}$ data (black) are drawn from an uncorrelated Normal distribution. (Bottom) The true $d_i^{(B)}$ data (black) are computed using $\mathbf{d}^{(B)} = \mathbf{Gm}$, where $\mathbf{m}$ is the true model parameters as shown in Fig. 2 and where $\mathbf{G}$ depends on the true $\mathbf{d}^{(A)}$. The predicted $\mathbf{d}^{(B)}$ closely match the true data.

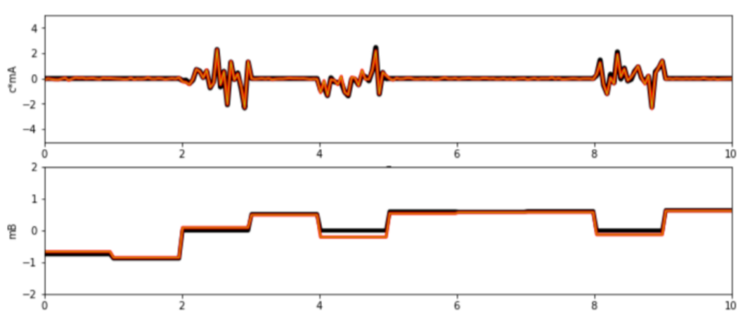

Fig. 2. The two parts of the model, $m_i^{(A)}$ (top) and $m_i^{(B)}$ (bottom), as a function of position, z_i. The synthetic true model (black) was chosen randomly. In each of the ten geographical regions, either $m_i^{(B)}$ is zero (in which case $m_i^{(A)}$ is drawn from an uncorrelated Normal distribution), or $m_i^{(B)}$ is non-zero (in which case $m_i^{(A)}$ is zero). The estimated model (red) is close to the true model.

The reweighting method is described in Chapter 8 of Menke, W., Geophysical Data Analysis: Discrete Inverse Theory, Fourth Edition, Elsevier, pp 350, 2018, ISBN: 9780128135556.