Thoughts on Differences between Fréchet Derivatives

Bill Menke, May 26, 2023

Although there are many kinds of "Fréchet derivative" (or "Sensitivity Kernels")[1], these works are often used in geophysics without qualification. The reader is assumed to know which "Fréchet derivative" or "Sensitivity Kernel" is being referred to. But consider an ordinary "derivative", as it appears in the sentence "I finally succeed in working out a formula for the derivative". The reader is completely in the dark of both the y and the x of dy/dx. Nevertheless, the identity of the two variables is important: dy/dx is not the same and du/dv! Very significant misunderstandings can occur when the different kinds of Fréchet derivative (or Sensitivity Kernels) are not distinguished.

As an example, consider a geodynamics problem in which an ice load causes deformation in a viscous Earth model, resulting in changes in relative sea leave.

The Fréchet derivative, $G(\theta_0, \varphi_0, t_0 | r, \theta, \varphi)$, quantifies the change in relative sea level, $s(\theta_0, \varphi_0, t_0)$, at latitude, θ_0, longitude, φ_0, and time, t_0, due to a point-perturbation, $\delta\mu(r, \theta, \varphi)$ in viscosity (where r is radius) with respect to a reference model.

$$G(\theta_0, \varphi_0, t_0 | r, \theta, \varphi) \equiv \frac{\delta s(\theta_0, \varphi_0)}{\delta\mu(r, \theta, \varphi)} \tag{1}$$

Here, a specific and known ice load history has been presumed. This is the Fréchet derivative (or sensitivity kernel) of relative sea level with respect to viscosity".

Relative sea level at a particular point, $(\theta_0, \varphi_0, t_0)$, is determined by integration of the product of the Fréchet derivative and the viscosity perturbation, $\delta\mu$, over the (r, θ, φ) Earth volume:

$$s(\theta_0, \varphi_0, t_0) = \big(G(\theta_0, \varphi_0,, t_0 | r, \theta, \varphi), \delta\mu(r, \theta, \varphi)\big)_{r,\theta,\varphi} \tag{2}$$

Here, $(a, b)_x \equiv \int ab \, dV_x$ is the inner product over the volume, V_x.

Let the viscosity perturbation, $\delta\mu$, be proportional to shear wave velocity perturbation, δv (determined, say, via geo-tomography), with the proportionally factor, $\beta(r)$, crudely dependent on depth via an upper mantle factor, β_u, and a lower mantle factor, β_l.

$$\delta\mu(r, \theta, \varphi) = \begin{cases} \beta_u \delta v(r, \theta, \varphi) & r \geq r_M \\ \beta_l \delta v(r, \theta, \varphi) & r < r_M \end{cases} \tag{3}$$

By substitution, sea level at a particular point, $(\theta_0, \varphi_0, t_0)$, is:

$$\begin{aligned} s(\theta_0, \varphi_0, t_0) = {} & \beta_u \big(G(\theta_0, \varphi_0, t_0 | r, \theta, \varphi), \delta v(r, \theta, \varphi)\big)_{r \geq r_M, \theta, \varphi} \\ & + \beta_l \big(G(\theta_0, \varphi_0, t_0 | r, \theta, \varphi), \delta v(r, \theta, \varphi)\big)_{r < r_M, \theta, \varphi} \end{aligned} \tag{4}$$

where the inner products are now over a limited range of radius.

Let the sensitivity kernel derivatives, $H_u(\theta_0, \varphi_0)$ and $H_l(\theta_0, \varphi_0)$ quantify the change in relative sea level, $s(\theta_0, \varphi_0, t_0)$, at latitude, θ_0, longitude, φ_0, and time, t_0, with respect to a change in the proportionality factors, β_u and β_l.

$$H_u(\theta_0, \varphi_0) \equiv \frac{\partial s(\theta_0, \varphi_0)}{\partial \beta_u} = \big(G(\theta_0, \varphi_0, t_0 | r, \theta, \varphi), \delta v(r, \theta, \varphi)\big)_{r \geq r_M, \theta, \varphi}$$

$$H_h(\theta_0, \varphi_0) \equiv \frac{\partial s(\theta_0, \varphi_0)}{\partial \beta_l} = \big(G(\theta_0, \varphi_0, t_0 | r, \theta, \varphi), \delta v(r, \theta, \varphi)\big)_{r < r_M, \theta, \varphi}$$

(5)

Now suppose that image $I_1(\theta, \varphi)$ depicts $G(\theta_0, \varphi_0, t_0 | r_H, \theta, \varphi)$ at fixed $(\theta_0, \varphi_0, t_0, r_H)$ and that $I_2(\theta, \varphi)$ depicts $H_u(\theta, \varphi)$. These two images are of different quantities and cannot be presumed to be the same. The (θ, φ) in $I_1(\theta, \varphi)$ is the point at the $r = r_H$ surface where the viscosity perturbation is being applied. The (θ, φ) in $I_1(\theta, \varphi)$ is the point at the Earth's surface where the relative sea level is being measured. Furthermore, $I_2(\theta, \varphi)$ involves a specific pattern of a viscosity variation (though the choice of the shear velocity perturbation), whereas $I_1(\theta, \varphi)$ is independent of this pattern.

[1]Distinction between the words "Fréchet derivative" and "Sensitivity Kernels.

Consider a function, $\mathbf{y}(\mathbf{x})$ where $\mathbf{y}$ is a vector of length, N, and $\mathbf{x}$ is a vector of length, M. The partial derivative, $\partial y_i / \partial x_j$, relates a perturbation, $\Delta \mathbf{y}$, to a perturbation, $\Delta \mathbf{x}$, about a reference value, $\mathbf{x}_0$, via

$$\Delta y_i = \sum_{j=1}^{M} \left. \frac{\partial y_i}{\partial x_j} \right|_{\mathbf{x}_0} \Delta x_j$$

(6)

A Fréchet derivative, $\delta y / \delta x$, is a generalization of this partial derivative to functions, $y(t)$ and $x(t)$. It relates a perturbation, $\delta y(t)$, to a perturbation, $\delta x(t)$ about a reference value, $x_0(t)$, via

$$\delta y(t) = \int \left. \frac{\delta y}{\delta x} \right|_{x_0} \delta x(t')\, dt'$$

(7)

When $\mathbf{x}$ is time series of length, M, and uniform spacing, Δx, Eq. (7) is just the limiting case of Eq. (6), when $M \to \infty$ and $\Delta x \to 0$; that is, of the time series becoming more and more finely sampled. What was a discrete index, j, in Eq. (6) has become a continuous variable, t', in Eq. (7).

The term sensitivity kernel applies to any quantity that relates a perturbation of in one variable to a perturbation in another through a linear rule. Thus, both

$$G_{ij} \equiv \left. \frac{\partial y_i}{\partial x_j} \right|_{\mathbf{x}_0} \quad \text{and} \quad G(t|t') \equiv \left. \frac{\delta y}{\delta x} \right|_{x_0}$$

(8)

are sensitivity kernels, because

$$\Delta y_i = \sum_{j=1}^{M} G_{ij}\, \Delta x_j \equiv \mathbf{G}\Delta\mathbf{x} \text{ and}$$

$$\delta y(t) = \int G(t|t')\, \delta x(t')\, dt' \equiv \big(G(t|t')\, \delta x(t')\big)_{t'} \tag{9}$$

are linear rules. However, only $G(t|t')$ is a Fréchet derivative.