Should Glacial Isostatic Calculation Consider Viscosity to be Advecting?
Bill Menke, June 6, 2023

(1) I am concerned here with the viscosity term in the Navier Stokes equation, in the case where the viscosity is spatially-heterogenous and dragged along by the flow. In the coupled temperature – flow problem, the dragging of viscosity is handled naturally by its dependence on temperature. Viscosity advects because the temperature field that creates it advects. But what about a glacial loading problem in which temperature is not included? My belief is that viscosity must still undergo advection. The viscosity term in the Navier Stokes equations should be thought of as $\mu(\mathbf{x},t)$ where t is the "current time". However, the viscosity that one wants to know is the "initial viscosity", $\mu^{(I)}(\mathbf{x}) \equiv \mu(\mathbf{x},t=0)$. So, I think this distinction needs to be made when one sets up the inverse problem, because the flow field past a stationary high-viscosity body with be very significantly perturbed by it (as in Stokes flow around a sphere), but the flow field containing a moving high-viscosity body less so.

(2) Basic equations

Incompressible flow

$$v_{j,j} = 0$$

Viscous rheology

$$\tau_{ij} = \mu\left(v_{i,j} + v_{j,i}\right) - p$$

Momentum-conservation (which I believe is correct even when viscosity, μ, is a function of time.

$$\rho\frac{\partial v_i}{\partial t} + v_{i,j}v_j = \tau_{ij,j} = \mu\left(v_{i,jj} + v_{j,ji}\right) + \mu_{,j}\left(v_{i,j} + v_{j,i}\right) - p_{,i}$$

Momentum-conservation given incompressible flow

$$\rho\frac{\partial v_i}{\partial t} + \rho v_{i,j}v_j = \mu v_{i,jj} + \mu_{,j}\left(v_{i,j} + v_{j,i}\right) - p_{,i}$$

Incompressible advection (viewing μ as a linear function of temperature, a conserved advecting quantity).

$$\frac{\partial \mu}{\partial t} + v_j\mu_{,j} = 0$$

(3) One could solve the coupled problem in which one explicitly advects viscosity. However, in glacial isostatic problems, the overall flow distance is small. In this case, one might be able to use a low order approximation. Considering viscosity a function of time, for small times

Viscosity and its gradient at time, $t = 0$

$$\mu^{(I)}(\mathbf{x}) \equiv \mu(\mathbf{x},t=0) \quad \text{and} \quad \frac{\partial \mu^{(I)}}{\partial t} \equiv \left.\frac{\partial \mu}{\partial t}\right|_{\mathbf{x},t=0} \quad \text{and} \quad \mu_{,i}^{(I)}(\mathbf{x}) \equiv \mu_{,i}(\mathbf{x},t=0)$$

Taylor's theorem to get viscosity at a small, later time

$$\mu(\mathbf{x},t) = \mu^{(I)}(\mathbf{x}) + \frac{\partial \mu^{(I)}}{\partial t}t$$

Advection equation at time, $t = 0$

$$\frac{\partial \mu^{(I)}}{\partial t} + v_k \mu_{,k}^{(I)} = 0$$

Insert into Taylor's theorem

$$\mu = \mu^{(I)} - v_k \mu_{,k}^{(I)} t$$

Take gradient

$$\mu_{,j} = \mu_{,j}^{(I)} - v_k \mu_{,jk}^{(I)} t - v_{k,j} \mu_{,k}^{(I)} t$$

Insert into momentum-conservation plus incompressible

$$\rho \frac{\partial v_i}{\partial t} + v_{i,j} v_j = \left[\mu^{(I)} - v_k \mu_{,k}^{(I)} t\right] v_{i,jj} + \left[\mu_{,j}^{(I)} - v_k \mu_{,jk}^{(I)} t - v_{k,j} \mu_{,k}^{(I)} t\right] \left(v_{i,j} + v_{j,i}\right) - p_{,i}$$

Note that the terms explicitly containing time t represent a correction for the advection of the material.

(4) If now one considers a point heterogeneity, $\mu^{(I)}(\mathbf{x}) = m\delta(\mathbf{x} - \mathbf{x}_H)$, the partial derivative of the differential operator with respect to m will contain $t\delta$, $t\delta_{,k}$ and $t\delta_{,jk}$.