Discussion of Units in 149: Effective Pressure Diffusivity of a Long, Thin Pipe

Bill Menke, October 1, 2023 after a discussion with Einat Lev

1. The pressure p has units of N/m^2, so the equation $\left(\frac{\partial}{\partial t} + \kappa \frac{\partial^2}{\partial x^2}\right) p = f$ has units of N/sm^2. Consequently, f has units of N/sm^2 and κ has units of m^2/s. Note that κt has units of m^2.
2. The Green Function equation integral is $p(x,t) = \iint g(x, x_0, t, t_0)\, f(x_0, t_0)\, dxdt$. As p has units of N/m^2, f has units of N/sm^2, and $dxdt$ has units of sm, g has units of
$$\frac{N}{m^2} = g\, \frac{N}{sm^2}\ \ sm \quad \text{so } g \text{ has units of} \quad \frac{N}{m^2}\frac{sm^2}{N}\frac{1}{sm} = \frac{1}{m}$$
3. The Green function equation $\left(\frac{\partial}{\partial t} + \kappa \frac{\partial^2}{\partial x^2}\right) g(x,0,t,0) = \delta(x)\delta(t)$ has solution
$$g(x,0,t,0) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\kappa t}} \exp\left\{-\frac{1}{2}\frac{x^2}{2\kappa t}\right\}$$
This solution requires $x^2/\kappa t$ to be dimensionless, which it is. It also agrees with the previous result that g has units of $1/m$.
4. For the force $f(x,t) = E_0 \delta(x)\delta(t)$, the solution is $p = E_0 g(x,0,t,0)$. The constant E_0 has units
$$\frac{N}{sm^2} = E_0 \frac{1}{sm} \quad \text{so } E_0 \text{ has units of} \quad \frac{N}{sm^2}\ sm = \frac{N}{m^2}\ m$$
This can be interpreted as a pressure, say of p_0 over an interval, say Δx; that is, $E_0 = p_0 \Delta x$. Although this superficially looks like an energy per unit area, it is actually a *moment* per unit area.
5. The original derivation defines a quantity $\beta = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$. Thus, β has units of reciprocal pressure, that is $\frac{m^2}{N}$.
6. The original derivation defines a quantity k such that velocity $v = -k\nabla p$. The units of k are
$$\frac{m}{s} = k \frac{1}{m}\frac{N}{m^2} \quad \text{so } k \text{ has units of} \quad \frac{m}{s} m \frac{m^2}{N} = \frac{m^4}{Ns}$$
7. The original derivation defines $\kappa = k/\beta$, which has units $\frac{m^4}{Ns}\frac{N}{Nm^2} = \frac{m^2}{s}$ which agrees with the units in (1), above.
8. Wikipedia give the Hagen–Poiseuille equation as $\Delta p = 8\mu L Q/\pi R^4$, where L is pipe length, R is pipe radius and Q is volumetric flow rate. Substituting $Q = v\pi R^2$ yields
$$\Delta p = \frac{8\mu L Q}{\pi R^4} = \frac{8\mu L v \pi R^2}{\pi R^4} = \frac{8\mu L v}{R^2} \quad \text{or} \quad \frac{R^2}{8\mu}\frac{\Delta p}{L} = v$$
As in the original derivation, we identify $k = R^2/8\mu$.

9. According to Wikipedia, μ has units of Ns/m^2. Thus, k has units of

$$m^2\frac{m^2}{Ns} = \frac{m^4}{Ns}$$

Which agrees with (6), above.

10. According to (4), above, the pressure corresponding to $f(x,t) = E_0\delta(x)\delta(t)$ is $p = E_0 g(x,0,t,0)$. According to (6), above, velocity is related to pressure via $v = -k\,\frac{\partial p}{\partial x}$. The Green function in (3) implies

$$v = -kE_0\frac{\partial}{\partial x}g(x,0,t,0) = -\frac{kE_0}{\sqrt{2\pi}}\frac{1}{\sqrt{2\kappa t}}\frac{\partial}{\partial x}\exp\left\{-\frac{1}{2}\frac{x^2}{2\kappa t}\right\}$$

$$= \frac{kE_0}{\sqrt{2\pi}}\frac{1}{\sqrt{2\kappa t}}\frac{x}{2\kappa t}\exp\left\{-\frac{1}{2}\frac{x^2}{2\kappa t}\right\}$$

11. The units of the right-hand side of the previous equation are

$$k\;E_0\frac{1}{\sqrt{\kappa t}}\frac{x}{\kappa t} = \;\frac{m^4}{Ns}\;\frac{Nm}{m^2}\frac{1}{m}\frac{m}{m^2} = \frac{Nm^6}{Nm^5 s} = \frac{m}{s}$$

Which agrees with the units of velocity, m/s, on left-hand side.