# On Reconciling Ray Amplitudes and Energy Fluxes

By Bill Menke, December 20, 2023

In ray theory for the acoustic (scalar) wave equation $c^2\nabla^2 p - \ddot{p} = 0$, the wavefield is parameterized as pressure $p(\mathbf{x}, t) = A(\mathbf{x}) \exp\{i\omega(T(\mathbf{x}) - t)\}$. Here $T$ is travel time and $A$ is amplitude. The material velocity $c(\mathbf{x})$ is variable but the density $\rho$ is constant. (The above wave equation is only correct for a constant density).

First, we define a quantity $E$ that will later turn out to be energy flux in the ray direction.

(1) $E = \rho^{-1}c^{-1}A^2$ or $A^2 = \rho c E$

In ray theory, the travel time $T$ and the ray direction $\mathbf{t}$ are related by:

(2) $c\nabla T = \mathbf{t}$ or $\nabla T = c^{-1}\mathbf{t}$ (I have checked this against sources on the web).

In ray theory, the transport equation for amplitude $A$ and travel time $T$ is

(3) $2\nabla A \cdot \nabla T = -A\nabla^2 T$ (I have checked this against sources on the web).

Note the identify

(4) $\nabla A^2 = 2A\nabla A$

Multiplying (3) by $A$ and substituting identify (4) yields

(5) $\nabla A^2 \cdot \nabla T = -A^2\nabla^2 T = A^2\nabla \cdot \nabla T$

Substituting (1) and (2) into (5) yields

(6) $\nabla(\rho c E) \cdot (c^{-1}\mathbf{t}\ ) = -(\rho c E)\nabla \cdot (c^{-1}\mathbf{t})$

Canceling the constant $\rho$ and apply the chain rule yields:

(7) $[c\nabla E + E\nabla c] \cdot (c^{-1}\mathbf{t}\ ) = -(cE)[c^{-1}\nabla \cdot \mathbf{t} + \mathbf{t} \cdot \nabla c^{-1}]$

Substituting $\nabla c^{-1} = -c^{-2}\nabla c$ into (7) yields

(8) $[c\nabla E + E\nabla c] \cdot (c^{-1}\mathbf{t}\ ) = -(cE)[c^{-1}\nabla \cdot \mathbf{t} - c^{-2}\mathbf{t} \cdot \nabla c\ ]$

Multiplying out (8) yields

(7) $\nabla E \cdot \mathbf{t} + c^{-1}E\nabla c \cdot \mathbf{t} = -E\nabla \cdot \mathbf{t} + c^{-1}E\nabla c \cdot \mathbf{t}$

Canceling the term $c^{-1}E\nabla c \cdot \mathbf{t}$ from both sides of the equation yields

(8) $\nabla E \cdot \mathbf{t} \ = -E\nabla \cdot \mathbf{t}$

Now consider the vector quantity $\mathbf{f} = E\mathbf{t}$. The equation $\nabla \cdot \mathbf{f} = 0$ can be written

(9) $\nabla \cdot \mathbf{f} = \nabla \cdot (E\mathbf{t}) = E\nabla \cdot \mathbf{t} + \nabla \mathrm{E} \cdot \mathbf{t} = 0$

Note that (9) is the same as (8). Thus, $\mathbf{f} = E\mathbf{t}$ is a conserved flux. We can also write (8) as

(10) $\frac{\nabla E}{E} \cdot \mathbf{t} \ = -\nabla \cdot \mathbf{t}$

That is, the fractional decrease in $E$ in the ray direction equals the divergence of the rays (which defines the concept of geometrical spreading).

The units of $E = \rho^{-1}c^{-1}A^2$ are energy flux; that is, $J/m^2s$

(11) $\frac{m^3}{kg}\frac{s}{m}\left(\frac{N}{m^2}\right)^2 = \frac{m^3}{kg}\frac{s}{m}\frac{N}{m^2}\frac{kg\ m}{s^2m^2} = \frac{N\ kg\ m^4\ s}{kg\ m^5\ s^2} = \frac{N{-}m}{m^2s} = \frac{J}{m^2s}$

Hence, **f** is energy flux. Ray theory is consistent with conservation of energy.

However, consider this quotation from Aki and Richards, Quantitative Seismology 2nd Ed, 2009, Section 5.1, p.121-122:

"It follows that the flux rate of energy transmission in a plane wave (i.e., the amount of energy transmitted per unit time across unit area normal to the direction of propagation) is $\rho\alpha\dot{u}^2$ for P-waves and $\rho\beta\dot{u}^2$ for S-waves. We have proved this result only for plane waves in homogeneous media, and it is a "local" property, depending on material properties and on the planar nature of the wave only at the point at which the flux rate is evaluated. We can therefore expect that flux rates are still given approximately by $\rho\dot{u}^2$ times the propagation velocity for the case of slightly curved wavefronts in a medium with some spatial fluctuation in material properties."

It seems that energy flux is being defines as $E = \rho c A^2$ and not as $E = \rho^{-1}c^{-1}A^2$ as in (1). However, the difference is due to the fact that the Aki and Richards are using particle velocity and not pressure. We should really write the Aki & Rickards formula as $E = \rho c \dot{U}^2$, where $\dot{U}^2$ is the amplitude of the particle velocity.

In an acoustic medium, pressure $p$ is related to particle displacement **u** by

(12) $p = \lambda \nabla \cdot \mathbf{u}$ with $\lambda = \rho c^2$

Writing pressure as $A\exp(ikx - i\omega t)$ and displacement as $U\exp(ikx - i\omega t)$, with $c = \omega/k$, and $U = \dot{U}/\omega$, (12) becomes

(13) $A = \rho c^2 \frac{k}{\omega}\dot{U} = \rho c \dot{U}$

Thus

(14) $E = \rho^{-1}c^{-1}A^2 = \rho^{-1}c^{-1}\rho^2c^2\dot{U}^2 = \rho c \dot{U}^2$

which matches Aki and Richards. One way to think about amplitude sensitivity is to write the perturbation in amplitude $dA = \left.\frac{\partial A}{\partial E}\right|_c dE + \left.\frac{\partial A}{\partial c}\right|_E dc$. Here, the term represents geometric spreading and the second "local energy conservation".

Note that if one is using the amplitude $\dot{U}$ of particle velocity, then at constant energy flux, $\dot{U}$ *decreases* when $c$ increases:

(15) $dE = 0 = \rho\dot{U}^2dc + 2\rho c\dot{U}d\dot{U}$ or $\frac{1}{\dot{U}}\frac{\partial \dot{U}}{\partial c} = -\frac{1}{2c}$

Whereas, if one is using the amplitude $A$ of pressure, then at constant energy flux, $A$ *increases* when $c$ increases:

(16) $dE = 0 = -\rho^{-1}c^{-2}A^2dc + \rho^{-1}c^{-1}2AdA$ or $\frac{1}{A}\frac{\partial A}{\partial c} = \frac{1}{2c}$

It is possible to construct a "scalar displacement" version of the scalar wave equation that is exactly analogous to the pressure equation; that is $c^2\nabla^2 u - \ddot{u} = 0$. However, there is a subtle difference. In order to derive this equation from the full equation $\rho\ddot{u}_i = \left(c_{ijpq}u_{p,q}\right)_{,j}$ (where $c_{ijpq}$ are the elastic parameters), you must assume $\left(c_{ijpq}u_{p,q}\right)_{,j} = c_{ijpq}u_{p,qj}$. That is that the elastic parameters must be constant and only the density $\rho$ may vary spatially. This change affects the way that derivatives are taken at (7). When this difference is taken into account, the correct flux is $E = \rho c\dot{U}^2$, as in Aki and Richards. This is shown as follows.

Suppose $E = \rho c\dot{U}^2 = \rho^{½}\gamma^{½}\dot{U}^2$ with $c = \gamma^{½}\rho^{-½}$ and $\gamma$ a constant, so $\dot{U}^2 = \rho^{-½}\gamma^{-½}E$. I am using $\gamma$ as an placeholder for $\lambda$ (in the acoustic case), $\lambda + 2\mu$ in the P wave case, and $\mu$ in the shear wave case. Then (6) becomes

(17) $\nabla\left(\rho^{-½}\gamma^{-½}E\right)\cdot\left(c^{-1}\mathbf{t}\ \right) = -\left(\rho^{-½}\gamma^{-½}E\right)\nabla\cdot\left(c^{-1}\mathbf{t}\right)$

Cancelling the constant $\gamma^{-½}$

(18) $\nabla\left(\rho^{-½}E\right)\cdot\left(c^{-1}\mathbf{t}\ \right) = -\left(\rho^{-½}E\right)\nabla\cdot\left(c^{-1}\mathbf{t}\right)$

Apply chain rule to the l.h.s.

(19) $\left[\rho^{-½}\nabla E - ½E\rho^{-½}\rho^{-1}\nabla\rho\right]\cdot\left(c^{-1}\mathbf{t}\ \right) = -\left(\rho^{-½}E\right)\nabla\cdot\left(c^{-1}\mathbf{t}\right)$

Canceling the factor of $\rho^{-½}$

(20) $\left[\nabla E - ½E\rho^{-1}\nabla\rho\right]\cdot\left(c^{-1}\mathbf{t}\ \right) = -E\nabla\cdot\left(c^{-1}\mathbf{t}\right)$

Now note the identify

(21) $\rho^{-1}\nabla\rho = \gamma\rho^{-1}\nabla(\rho/\gamma) = c^2\nabla(c^{-2}) = -2c^2c^{-3}\nabla\mathrm{c} = -2c^{-1}\nabla\mathrm{c}$

Insert the identity (21) into (20)

(22) $\left[\nabla E + c^{-1}\nabla\mathrm{c}\right]\cdot\left(c^{-1}\mathbf{t}\ \right) = -E\nabla\cdot\left(c^{-1}\mathbf{t}\right)$

Applying the chain rule to the r.h.s.

(23) $\left[\nabla E + c^{-1}\nabla\mathrm{c}\right]\cdot\left(c^{-1}\mathbf{t}\ \right) = -Ec^{-1}\nabla\cdot\mathbf{t} + c^{-2}\nabla\mathrm{c}\cdot\mathbf{t}$

Cancel terms $c^{-2}\nabla\mathrm{c}\cdot\mathbf{t}$ and divide by $c^{-1}E$

(24) $\frac{\nabla E}{E} = -\nabla\cdot\mathbf{t}$

This derivation achieves the same energy flux equation as in (10), even though the definition of energy flux is different than (1). Counterintuitively, even though the pressure and displacement scalar equations have exactly the same form, the formulas for energy flux are different. Ray theory automatically "adjusts" to this difference, and in both cases is compatible with conservation of energy.

Thus, the only "energy physics" in ray theory is: (A) energy flux is parallel to the ray; and (B) energy is conserved. Mathematically, (A) is represented as $\mathbf{f} = E\mathbf{t}$ and (B) as $\nabla\cdot\mathbf{f} = 0$.

We now investigate the conditions that the transport equation imposes on the conserved energy flux. Let $A^2 = b_0 b(\mathbf{x})E$ and $c = f_0 f(\mathbf{x})$. The transport equation becomes

(25) $\frac{\nabla(b_0 b(\mathbf{x})E)}{b_0 b(\mathbf{x})E} \cdot \mathbf{t} = -c \ \nabla \cdot \left(\frac{\mathbf{t}}{c}\right)$

Applying the chain rule

(26) $\frac{\nabla E}{E} \cdot \mathbf{t} + b^{-1}\nabla b \cdot \mathbf{t} = -\nabla \cdot \mathbf{t} + c^{-1}\nabla \mathrm{c} \cdot \mathbf{t} = -\nabla \cdot \mathbf{t} + f^{-1}\nabla f \cdot \mathbf{t}$

Two terms cancel to leave the conservation equation only when

(27) $b^{-1}\nabla b = f^{-1}\nabla f$ or $\nabla \ln b = \nabla \ln f$

Which is satisfied only when $\ln b = \ln f + constant$ or $b \propto f$. As we have already allowed for multiplicative constants, the condition $b = f$ suffices. Thus, when $c = f_0 f(\mathbf{x})$, the transport equation requires that the energy flux have the form $E = A^2/b_0 f(\mathbf{x})$ where $f_0$ and $b_0$ must be determined by other conditions, including the units of $A$ and $c$ and the approximations made to the underlying differential equation. In the cases we have been discussing:

(28) Pressure case: $f_0 = 1, f = c$ and $b_0 = \rho_0$, so that $c = c$ a and $E = \rho_0^{-1}c^{-1}A^2$

(29) Velocity case: $f_0 = \gamma_0^{½}, f = \rho^{-½}$ and $b_0 = \gamma_0^{-½}$, so that $c = \gamma_0^{½}\rho^{-½}$ a and $E = \gamma_0^{½}\rho^{½}\dot{U}^2$

One final thought. When $E = \rho c \dot{U}^2 = \rho^{½}\gamma^{½}\dot{U}^2$ and $\gamma$ is constant, then at constant energy flux

(30) $dE = 0 = ½\rho^{-½}\gamma^{½}\dot{U}^2 d\rho + 2\rho^{½}\gamma^{½}\dot{U}d\dot{U}$

Rearranging, we find

(31) $\frac{1}{\dot{U}}\frac{d\dot{U}}{d\rho} = -\frac{1}{4\rho}$

Noting that

(32) $\frac{dc}{d\rho} = \frac{d}{d\rho}\left(\gamma^{½}\rho^{-½}\right) = -½\gamma^{½}\rho^{-3/2}$

Combining (31) and (32) we obtain

(33) $\frac{1}{\dot{U}}\frac{d\dot{U}}{dc} = \frac{1}{\dot{U}}\frac{d\dot{U}}{d\rho}\frac{dc}{d\rho} = \frac{1}{4\rho}\frac{2\rho^{3/2}}{\gamma^{½}} = \frac{1}{2}\frac{\rho^{½}}{\gamma^{½}} = \frac{1}{2c}$

So, in the Aki and Richards case, the sign of the sensitivity $\frac{1}{\dot{U}}\frac{d\dot{U}}{dc}$ depends on whether the density or the Lame parameter is assumed constant.