Maximum Likelihood Applied to Binary Time Series

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Consider the binary time series **x** where each elements x_i takes only the values 0 and 1. We consider the case where the statistics of the time series are defined by the conditional pmf $P(x_{i+1}|x_i)$. We assert that the time series has no directional asymmetry, so that $P(x_i|x_{i+1}) = P(x_{i+1}|x_i)$, and limited memory, in the sense that $P(x_{i+1}|x_i, x_{i-1}, x_{i-2}, \cdots) = P(x_{i+1}|x_i)$. A simple parameterization is:

$$P(x_{i+1}|x_i): \quad \begin{array}{ccc} x_i \backslash x_{i+1} & 0 & 1 \\ 0 & a_1 & (1-a_1) \\ 1 & (1-a_2) & a_2 \end{array} \tag{1}$$

Bayes theorem can be used to calculate the joint pmf, $P(x_i, x_{i+1})$:

$$P(x_i, x_{i+1}) = P(x_i|x_{i+1})P(x_{i+1}) = P(x_{i+1}|x_i)P(x_i) \tag{2}$$

We rearrange to

$$\frac{P(x_{i+1}|x_i)}{P(x_i|x_{i+1})} = \frac{P(x_{i+1})}{P(x_i)} \tag{3}$$

and the sum over values of x_{i+1}

$$D(x_i) \equiv \sum_{x_{i+1}} \frac{P(x_{i+1}|x_i)}{P(x_i|x_{i+1})} = \sum_{x_{i+1}} \frac{P(x_{i+1})}{P(x_i)} = \frac{1}{P(x_i)} \tag{4}$$

Substituting into (2) yields

$$P(x_i, x_{i+1}) = P(x_{i+1}|x_i)P(x_i) = \frac{P(x_{i+1}|x_i)}{D(x_i)} \tag{4}$$

In our case, x_i can take only the values $(0,1)$, so the sum in (4) becomes

$$\begin{aligned} D(x_i = 0) :&= \frac{P(x_{i+1} = 0|x_i = 0)}{P(x_i = 0|x_{i+1} = 0)} + \frac{P(x_{i+1} = 1|x_i = 0)}{P(x_i = 0|x_{i+1} = 1)} = \left(\frac{a_1}{a_1} + \frac{(1-a_1)}{(1-a_2)}\right) \\ D(x_i = 1) :&= \frac{P(x_{i+1} = 0|x_i = 1)}{P(x_i = 1|x_{i+1} = 0)} + \frac{P(x_{i+1} = 1|x_i = 1)}{P(x_i = 1|x_{i+1} = 1)} = \left(\frac{(1-a_2)}{(1-a_1)} + \frac{a_2}{a_2}\right) \end{aligned} \tag{5}$$

Inserting values for the conditionals yields:

$$P(x_i, x_{i+1}) = \frac{P(x_{i+1}|x_i)}{D(x_i)} : \begin{array}{ccc} x_i \backslash x_{i+1} & 0 & 1 \\ 0 & \left(\dfrac{a_1}{\left(1 + \dfrac{(1-a_1)}{(1-a_2)}\right)}\right) & \left(\dfrac{(1-a_1)}{\left(1 + \dfrac{(1-a_1)}{(1-a_2)}\right)}\right) \\ 1 & \left(\dfrac{(1-a_2)}{\left(\dfrac{(1-a_2)}{(1-a_1)} + 1\right)}\right) & \left(\dfrac{a_2}{\left(\dfrac{(1-a_2)}{(1-a_1)} + 1\right)}\right) \end{array} \tag{6}$$

which can be simplified to:

$$P(x_i, x_{i+1}) : \begin{array}{ccc} x_i \backslash x_{i+1} & 0 & 1 \\ 0 & \left(\dfrac{a_1(1-a_2)}{(1-a_1)+(1-a_2)}\right) & \left(\dfrac{(1-a_1)(1-a_2)}{(1-a_1)+(1-a_2)}\right) \\ 1 & \left(\dfrac{(1-a_1)(1-a_2)}{(1-a_1)+(1-a_2)}\right) & \left(\dfrac{a_2(1-a_1)}{(1-a_1)+(1-a_2)}\right) \end{array} \tag{7}$$

Because $P(x_i, x_{i+1})$ is symmetrical, row sums equal columns sums and the univariate pmf's are $P(x_{i+1}) = P(x_i)$. Their values are

$$P(x_i) = \sum_{x_{i+1}} P(x_i, x_{i+1}) : \begin{array}{cc} x_i & \\ 0 & \left(\dfrac{(1-a_2)}{(1-a_1)+(1-a_2)}\right) \\ 1 & \left(\dfrac{(1-a_1)}{(1-a_1)+(1-a_2)}\right) \end{array} \tag{8}$$

This result is abbreviated as

$$P(x_i) : \begin{array}{cc} x_i & \\ 0 & m_0 \\ 1 & m_1 \end{array} \tag{9}$$

Note the row sum is unity, as is required for a well-posed pmf. Note also when $a_1 = a_2$, $m_0 = m_1 = ½$ (as is required by symmetry under exchange of a_1 and a_2).

The probability of a particular time series **x** of length N can be computed by starting with the chain rule $P(\mathbf{x}) = P(x_1)P(x_2|x_1)P(x_3|x_2, x_1)P(x_4|x_3, x_2, x_1) \cdots P(x_N|x_{N-1}, \cdots x_1)$ and applying the limited memory principle. The result can be written either left to right or right to left:

$$P(\mathbf{x}) = \begin{cases} P(x_1)P(x_2|x_1)P(x_3|x_2)\cdots P(x_N|x_{N-1}) \\ P(x_1|x_2)P(x_2|x_3)\cdots P(x_{N-1}|x_N)P(x_N) \end{cases} \tag{10a,b}$$

These two forms are equal as long as the univariate pmf $P(x_i)$ is independent of i and the conditional pmf $P(x_{i+1}|x_i)$ is invariant under exchange x_{i+1} and x_i. Thus, (10a) can be used to compute a realization of **x** by first drawing x_1 from $P(x_1)$, then drawing x_2 from $P(x_2|x_1)$, then drawing x_3 from $P(x_3|x_2)$, and so forth. This process is reminiscent of an AR1 process.

As the time series $(x_i, x_{i+1}, x_{i+2}, \cdots x_{i+N-1})$ can be represented by an N-bit integer, the joint pmf can be written $P_I(I)$, where I is the integer.

$$P_I(I) = P(\mathbf{x}) \tag{11}$$

The joint P_I for a given I can be calculated by writing I as a sequence $(b_1, \cdots b_N)$ of 0s and 1s, initializing $P_I = P(b_1)$ and then performing a sequence of multiplies, in which P_0 is replaced by $P_0 P(b_2|b_1)$, and that result by $P_0 P(b_3|b_2)$ and so forth.

The conditional $A(j) = P\big(x_{i+j} = 1|x_i = 1\big)$, with $j \geq 0$ is reminiscent of an autocorrelation function. We note that $A(0) = 1$, and $A(1) = P(x_{i+1} = 1|x_i = 1)$. The values for $j > 1$ can be calculated by considering every possible bit string of the form $(1, b_2, \cdots, b_{N-1}, 1)$, calculating its probability (as above) and summing the probabilities. As the bit strings $(b_2, \cdots, b_{N-1})$ are $(N-2)$-bit integers, they can be derived from the sequence of integers $(0, \cdots, 2^{N-2} - 1)$. Analogous algorithms can be used for $P\big(x_{i+j} = 0|x_i = 0\big)$ and etc.

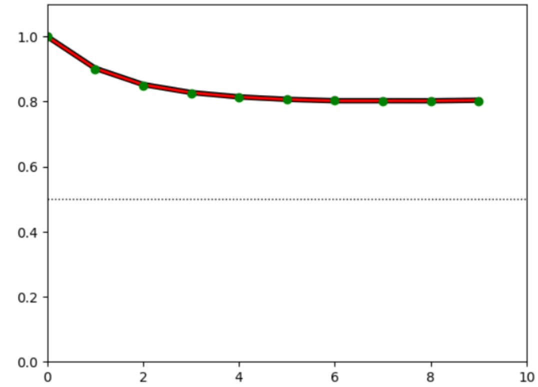
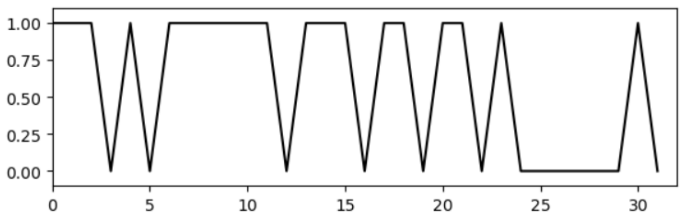
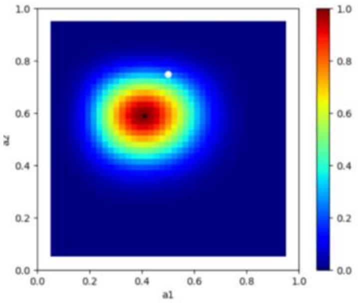
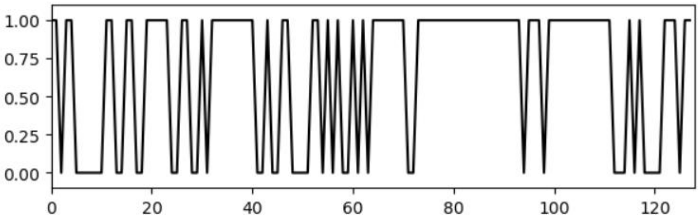
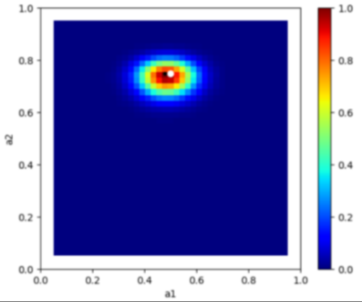
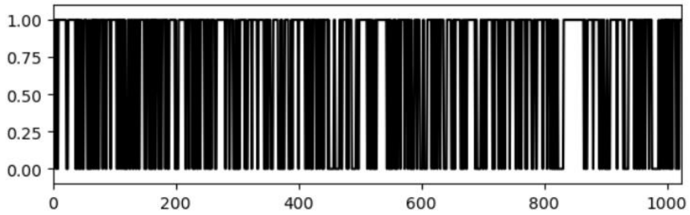
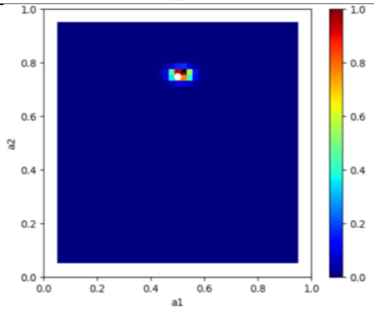


Fig 1. $A(j)$ as a function of j for $(a_1, a_2) = (0.6, 0.9)$, with theoretical (green); forward-stepping numerical calculation (red) and backward-stepping numerical (black) (for $N = 20{,}000$).

For $a_1 > ½$ and $a_2 > ½$, numerical experiments show that the conditional $P\big(x_{i+j} = 1|x_i = 1\big)$ decreases monotonically with j, from unity for $j = 0$, and approaches m_1 as j is increased. The behavior is reminiscent of an autocorrelation function.

Assuming that the prior pdf $P_A(a_1, a_2)$ is uniform, the posterior pdf is $P(a_1, a_2|\mathbf{x}_0) = P(\mathbf{x}_0|a_1, a_2)$. Then the maximum likelihood estimate of (a_1, a_2) is:

$$(a_1^{est}, a_2^{est}) = \underset{(a_1, a_2)}{\operatorname{argmax}} \log P(a_1, a_2|\mathbf{x}_0) \tag{12}$$

N	(A) time series $\mathbf{x}$	(B) Likelihood surface
32		
128		
1024		

Maximum likelihood estimation of (a_1, a_2). True values are $(0.50, 0.75)$ (white), estimated values (black), $P(a_1^{est}, a_2^{est}|\mathbf{x}_0)$ (colors).

A single N-element binary time series contains only N bits of information. Consequently, one need fairly large values of N (say $N > 100$) to achieve accurate estimates of (a_1, a_2).

Suppose that we denote the number of time that $P(x_1 = i)$ appears in the Eq. 10a as N_i and the number of time that $P(x_i = i|x_{i+1} = j)$ appears as N_{ij}. Then, the likelihood of a particular $\mathbf{x}$ is

$$P(a_1, a_2|\mathbf{x}) = [P(0)]^{N_0}[P(1)]^{N_1} \prod_{i=1}^{N_{00}} P(0|0) \prod_{i=1}^{N_{01}} P(0|1) \prod_{i=1}^{N_{10}} P(1|0) \prod_{i=1}^{N_{11}} P(1|1)$$

(13)

The log likelihood is then

$$L \equiv \log P(a_1, a_2|\mathbf{x}) = N_0 \log P(0) + N_2 \log P(1) +$$

$$N_{00} \log P(0|0) + N_{01} \log P(0|1) + N_{10} \log P(1|0) + N_{11} \log P(1|1)$$

(14)

The partial derivative with respect to a_i is

$$\frac{\partial L}{\partial a_i} = \frac{N_0}{P(0)}\frac{\partial P(0)}{\partial a_i} + \frac{N_1}{P(1)}\frac{\partial P(1)}{\partial a_i} +$$

$$\frac{N_{00}}{P(0|0)}\frac{\partial P(0|0)}{\partial a_i} + \frac{N_{01}}{P(0|1)}\frac{\partial P(0|1)}{\partial a_i} + \frac{N_{10}}{P(1|0)}\frac{\partial P(1|0)}{\partial a_i} + \frac{N_{11}}{P(1|1)}\frac{\partial P(1|1)}{\partial a_i}$$

(15)

Let $d \equiv [(1 - a_1) + (1 - a_2)]^{-1}$. Then,

$$d' \equiv \frac{\partial d}{\partial a_1} = \frac{\partial d}{\partial a_2} = [(1 - a_1) + (1 - a_2)]^{-2} = d^2$$

(16)

Differentiating Eq. (8) we obtain

$$\frac{\partial P(x)}{\partial a_1}: \begin{array}{cc} x & \\ 0 & \left((1 - a_2)d'\right) \\ 1 & (-d + (1 - a_1)d') \end{array} \quad \text{and} \quad \frac{\partial P(x)}{\partial a_2}: \begin{array}{cc} x & \\ 0 & (-d + (1 - a_2)d') \\ 1 & \left((1 - a_1)d'\right) \end{array}$$

(17)

And differentiating Eq. (7) we obtain

$$\frac{\partial P(x_i|x_{i+1})}{\partial a_1}: \begin{array}{ccc} x_i \backslash x_{i+1} & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{array} \quad \text{and} \quad \frac{\partial P(x_i|x_{i+1})}{\partial a_2}: \begin{array}{ccc} x_i \backslash x_{i+1} & 0 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{array}$$

(18)

These derivatives can be used in conjunction with the gradient descent method to determine (a_1^{est}, a_2^{est}), and their confidence intervals by estimating by sampling $P(a_1, a_2|\mathbf{x}_0)$ on a grid and estimating its width in the coordinate directions.

Reasonably well-tested Python functions:

```
def doPc(a1,a2):
# in my notation P(j|i) is Pc[i,j]
    b1 = 1.0-a1; # P(1|0)
    b2 = 1.0-a2; # P(1|0)
    Pc = np.array( [[a1,b1],[b2,a2]] );
    return Pc;

def doPind(Pc):
    a1 = Pc[0,0];
    b1 = 1.0 - a1;
    a2 = Pc[1,1];
    b2 = 1.0 - a2;
    Pind = np.zeros((2,1));
    d = b1 + b2;
    Pind[0,0] = b2/d;
    Pind[1,0] = b1/d;
    return Pind;

def doPjoint(Pc):
    a1 = Pc[0,0];
    b1 = 1.0 - a1;
    a2 = Pc[1,1];
    b2 = 1.0 - a2;
    Pjoint = np.zeros((2,2));
    d = b1 + b2;
    Pjoint[0,0] = a1*b2/d;
    Pjoint[0,1] = b1*b2/d;
    Pjoint[1,0] = Pjoint[0,1];
    Pjoint[1,1] = a2*b1/d;
    return Pjoint;

def doPI(Pc,Pind,I0,N):
    fmt = "{0:0%db}" % (N);
    s = fmt.format(I0);
    if( s[0] == "0" ):
        PI = Pind[0];
    else:
        PI = Pind[1];
    for j in range(1,N):
        s1 = s[j-1];
        s2 = s[j];
        if( (s1=="1") and (s2=="1") ):    # P(1|1)
            PI = PI*Pc[1,1];
        elif( (s1=="1") and (s2=="0") ):  # P(0|1);
            PI = PI*Pc[1,0];
        elif( (s1=="0") and (s2=="1") ):  # P(1|0)
            PI = PI*Pc[0,1];
        else:                             # P(0|0)
            PI = PI*Pc[0,0];
    return PI;
```

```
def dologPx(Pc,Pind,x0,N):
    logPc = np.log(Pc);
    logPind = np.log(Pind);
    if( x0[0,0] == 0 ):
        logPx = logPind[0];
    else:
        logPx = logPind[1];
    for j in range(1,N):
        x1 = x0[j-1,0];
        x2 = x0[j,0];
        if( (x1==1) and (x2==1) ):    # P(1|1)
            logPx = logPx+logPc[1,1];
        elif( (x1==1) and (x2==0) ):  # P(0|1);
            logPx = logPx+logPc[1,0];
        elif( (x1==0) and (x2==1) ):  # P(1|0)
            logPx = logPx+logPc[0,1];
        else:                             # P(0|0)
            logPx = logPx+logPc[0,0];
    return logPx;

def doA(Pc,L):
    A = np.zeros((L,1));
    A[0,0] = 1.0;
    A[1,0] = Pc[1,1];
    for K in range(3,L+1):
        Km2 = K-2;
        M = int(2**Km2);
        z = 0.0
        for i in range(M):
            fmt = "{0:0%db}" % (Km2);
            s = "1" + fmt.format(i) + "1";
            zz = 1.0
            for j in range(K-1):
                s1 = s[j];
                s2 = s[j+1]
                if( (s1=="1") and (s2=="1") ):    # P(1|1)
                    zz = zz*Pc[1,1];
                elif( (s1=="1") and (s2=="0") ):  # P(0|1);
                    zz = zz*Pc[1,0];
                elif( (s1=="0") and (s2=="1") ):  # P(1|0)
                    zz = zz*Pc[0,1];
                else:                             # P(0|0)
                    zz = zz*Pc[0,0];
            z = z + zz;
            A[K-1,0]=z;
    return A;

def dorealize(Pc,Pind,N):
    x = np.zeros((N,1),dtype=int);
    r = np.random.uniform(low=0.0, high=1.0);
    myp = Pind[1,0];
    if( r <= myp ):
        x[0,0]=1;
```

```
    else:
        x[0,0]=0;
    for i in range(1,N):
        p = x[i-1,0];
        myp = Pc[p,1];  # P(1|p)
        r = np.random.uniform(low=0.0, high=1.0);
        if( r <= myp ):
            x[i,0]=1;
        else:
            x[i,0]=0;
    return x;

def doItos(I,N):
    fmt = "{0:0%db}" % (N);
    s = fmt.format(I);
    return s;

def dostoI(s,N):
    I=0;
    for i in range(N):
        if( s[i] == "1" ):
            I = I + int(2**(N-i-1));
    return I;

def dostox(s,N):
    x = np.zeros((N,1));
    for i in range(N):
        if( s[i] == "1" ):
            x[i,0]=1;
    return x;

def doxtos(x,N):
    s = "";
    for i in range(N):
        if( x[i,0] == 0 ):
            s = s + "0";
        else:
            s = s + "1";
    return s

def Pderivs(Pc):
    Pind = doPind(Pc);
    a1 = Pc[0,0];
    a2 = Pc[1,1];
    d = 1.0/( (1.0-a1) + (1.0-a2) );
    dp = d**2;
    # univariate
    dP0da1 = (1.0-a2)*dp;
    dP1da1 = -d + (1.0-a1)*dp;
    dP0da2 = -d + (1.0-a2)*dp;
    dP1da2 = (1.0-a1)*dp;
    dPindda1 = gda_cvec( dP0da1, dP1da1 );
    dPindda2 = gda_cvec( dP0da2, dP1da2 );
```

```
    # conditional
    dPcda1 = np.array( [ [1.0, -1.0], [ 0.0, 0.0] ] );
    dPcda2 = np.array( [ [0.0,  0.0], [-1.0, 1.0] ] );
    return Pind, dPindda1, dPindda2, dPcda1, dPcda2;

def doLderivs(Pc,x0,N):
    Pind, dPindda1, dPindda2, dPcda1, dPcda2 = Pderivs(Pc);
    logPc = np.log(Pc);
    logPind = np.log(Pind);
    N0 = 0;
    N1 = 0;
    if( x0[0,0] == 0 ):
        N0 = N0 + 1;
    else:
        N1 = N1 + 1;
    N00 = 0;
    N01 = 0;
    N10 = 0;
    N11 = 0;
    for j in range(1,N):
        x1 = x0[j-1,0];
        x2 = x0[j,0];
        if( (x1==1) and (x2==1) ):    # P(1|1)
            N11 = N11 + 1;
        elif( (x1==1) and (x2==0) ):  # P(0|1);
            N01 = N01 + 1;
        elif( (x1==0) and (x2==1) ):  # P(1|0)
            N10 = N10 + 1;
        else:                         # P(0|0)
            N00 = N00 + 1;
    # L is log likelihood
    L = N0*logPind[0,0] + N1*logPind[1,0];
    L = L+N00*logPc[0,0]+N01*logPc[1,0]+N10*logPc[0,1]+N11*logPc[1,1];
    dLda1 = (N0/Pind[0,0])*dPindda1[0,0] + (N1/Pind[1,0])*dPindda1[1,0];
    dLda1 = dLda1 + (N00/Pc[0,0])*dPcda1[0,0] + (N01/Pc[1,0])*dPcda1[1,0];
    dLda1 = dLda1 + (N10/Pc[0,1])*dPcda1[0,1] + (N11/Pc[1,1])*dPcda1[1,1];
    dLda2 = (N0/Pind[0,0])*dPindda2[0,0] + (N1/Pind[1,0])*dPindda2[1,0];
    dLda2 = dLda2 + (N00/Pc[0,0])*dPcda2[0,0] + (N01/Pc[1,0])*dPcda2[1,0];
    dLda2 = dLda2 + (N10/Pc[0,1])*dPcda2[0,1] + (N11/Pc[1,1])*dPcda2[1,1];
    return L, dLda1, dLda2;
```