Damped Least Squares Data and Model Resolution Equal for Symmetric Data Kernel
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Let $\mathbf{m}$ be a length-M model parameter vector, $\mathbf{d}$ be a length-N data vector, $\mathbf{G}$ be a $N \times M$ data kernel matrix satisfying $\mathbf{Gm} = \mathbf{d}$ with uniform, uncorrelated covariance $\mathbf{C}_d = \sigma_d^2 \mathbf{I}$. Furthermore, suppose the prior information $\mathbf{m} = \mathbf{0}$ with uniform, uncorrelated covariance $\mathbf{C}_A = \sigma_A^2 \mathbf{I}$. Here, σ_d^2 and σ_A^2 are the variance of the observations and prior information, respectively. The Generalized least squares solution is achieved by solving

$$\mathbf{Fm} = \mathbf{h} \quad \text{with} \quad \mathbf{F} \equiv \begin{bmatrix} \sigma_d \mathbf{G} \\ \sigma_A \mathbf{I} \end{bmatrix} \quad \text{and} \quad \mathbf{f} \equiv \begin{bmatrix} \sigma_d \mathbf{d}^{obs} \\ \mathbf{0} \end{bmatrix}$$

by least squares. The result is called damped least squares, with damping parameter ε

$$\mathbf{m}^{est} \equiv [\mathbf{F}^T\mathbf{F}]^{-1}\mathbf{F}^T\mathbf{f} = [\mathbf{G}^T\mathbf{G} + \varepsilon^2\mathbf{I}]^{-1}\mathbf{G}^T\mathbf{d}^{obs} \equiv \mathbf{G}^{-g}\mathbf{d}^{obs} \ \text{ with } \ \varepsilon = \frac{\sigma_d}{\sigma_a}$$

Here, $\mathbf{G}^{-g}$ is the generalized inverse. The data and model resolution matrices are defined as

$$\mathbf{N}_G \equiv \mathbf{G}\mathbf{G}^{-g} \quad \text{and} \quad \mathbf{R}_G \equiv \mathbf{G}^{-g}\mathbf{G}$$

Now suppose that $\mathbf{G}$ has singular value composition

$$\mathbf{G} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^T$$

with $\mathbf{\Lambda}$ a diagonal matrix and $\mathbf{U}$ and $\mathbf{V}$ unary matrices (i.e., $\mathbf{V}^{-1} = \mathbf{V}^T$ and $\mathbf{U}^{-1} = \mathbf{U}^T$). Then

$$\mathbf{G}^T\mathbf{G} = \mathbf{V}\mathbf{\Lambda}\mathbf{U}^T\mathbf{U}\mathbf{\Lambda}\mathbf{V}^T = \mathbf{V}\mathbf{\Lambda}^2\mathbf{V}^T \quad \text{and} \quad \mathbf{I} = \mathbf{V}\mathbf{V}^T = \mathbf{V}^T\mathbf{V}$$

The data resolution is

$$\mathbf{N}_G = \mathbf{G}[\mathbf{G}^T\mathbf{G} + \varepsilon^2\mathbf{I}]^{-1}\mathbf{G}^T = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^T[\mathbf{V}\mathbf{\Lambda}^2\mathbf{V}^T + \varepsilon^2\mathbf{V}\mathbf{V}^T]^{-1}\mathbf{V}\mathbf{\Lambda}\mathbf{U}^T$$

$$= \mathbf{U}\mathbf{\Lambda}\mathbf{V}^T\mathbf{V}[\mathbf{\Lambda}^2 + \varepsilon^2\mathbf{I}]^{-1}\mathbf{V}^T\mathbf{V}\mathbf{\Lambda}\mathbf{U}^T = \mathbf{U}\mathbf{\Lambda}[\mathbf{\Lambda}^2 + \varepsilon^2\mathbf{I}]^{-1}\mathbf{\Lambda}\mathbf{U}^T = \mathbf{U}[\mathbf{\Lambda}^2 + \varepsilon^2\mathbf{I}]^{-1}\mathbf{\Lambda}^2\mathbf{U}^T$$

Here, we have used the facts that for any invertible matrix $\mathbf{M}$, $(\mathbf{V}\mathbf{M}\mathbf{V}^T)^{-1} = \mathbf{V}\mathbf{M}^{-1}\mathbf{V}^T$ (as can be seen from $\mathbf{V}\mathbf{M}\mathbf{V}^T\mathbf{V}\mathbf{M}^{-1}\mathbf{V}^T = \mathbf{V}\mathbf{M}\mathbf{M}^{-1}\mathbf{V}^T = \mathbf{V}\mathbf{V}^T = \mathbf{I}$) and that diagonal matrices commute. The model resolution is

$$\mathbf{R}_G = [\mathbf{G}^T\mathbf{G} + \varepsilon^2\mathbf{I}]^{-1}\mathbf{G}^T\mathbf{G} = [\mathbf{V}\mathbf{\Lambda}^2\mathbf{V}^T + \mathbf{V}\varepsilon^2\mathbf{I}\mathbf{V}^T]^{-1}\mathbf{V}\mathbf{\Lambda}^2\mathbf{V}^T$$

$$= \mathbf{V}[\mathbf{\Lambda}^2 + \varepsilon^2\mathbf{I}]^{-1}\mathbf{\Lambda}^2\mathbf{V}^T$$

We note that both $\mathbf{N}_G$ and $\mathbf{R}_G$ are symmetric, as $[\mathbf{M}^T\mathbf{S}\mathbf{M}]^T = \mathbf{M}^T\mathbf{S}\mathbf{M}$ for any symmetric matrix $\mathbf{S}$ (and diagonal matrices are symmetric). Furthermore, $\mathbf{U} = \mathbf{V}$ when $\mathbf{G}$ is symmetric, so the conditions that $\mathbf{N}_G = \mathbf{R}_G$ are that $N = M$ and $\mathbf{G}$ is symmetric.

Reference

Menke, W., Review of the Generalized Least Squares Method, Surveys in Geophysics 36, 1-25, 2014.