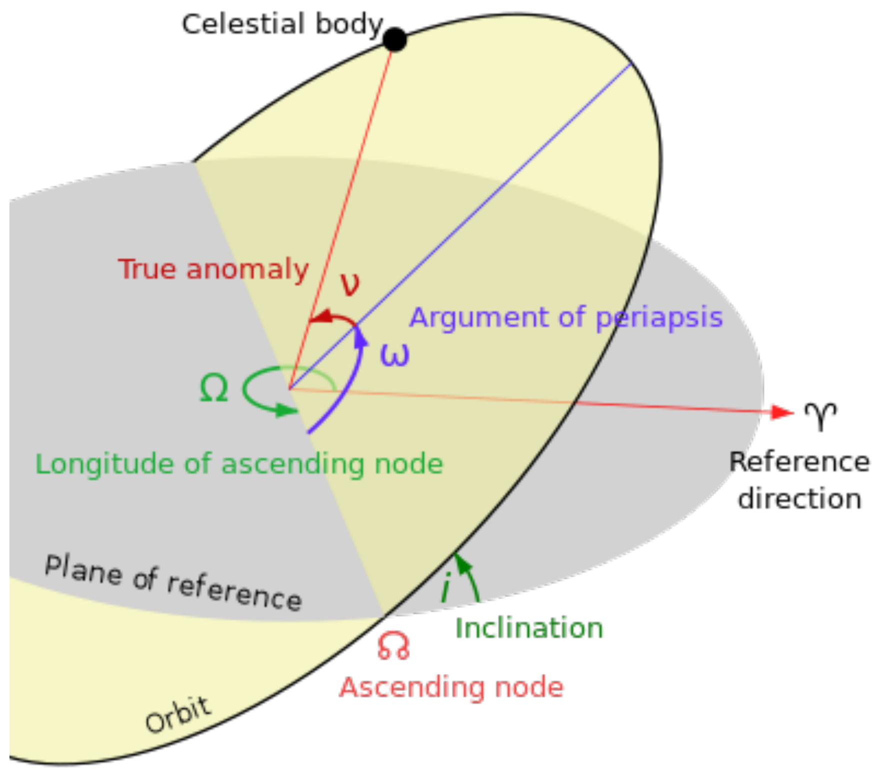
EESC 9945

Geodesy with the Global Positioning System

Class 3: *The GPS Constellation*

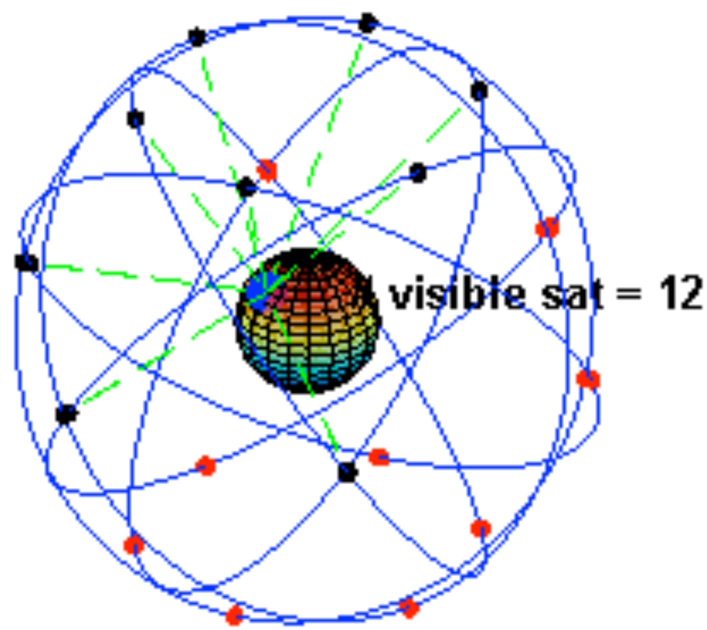
Review-Orbits

- Keplerian orbital parameters:
  - Semimajor axis
  - Eccentricity
  - Initial anomaly
  - Longitude of ascending node
  - Inclination
  - Argument of perigee



GPS Constellation

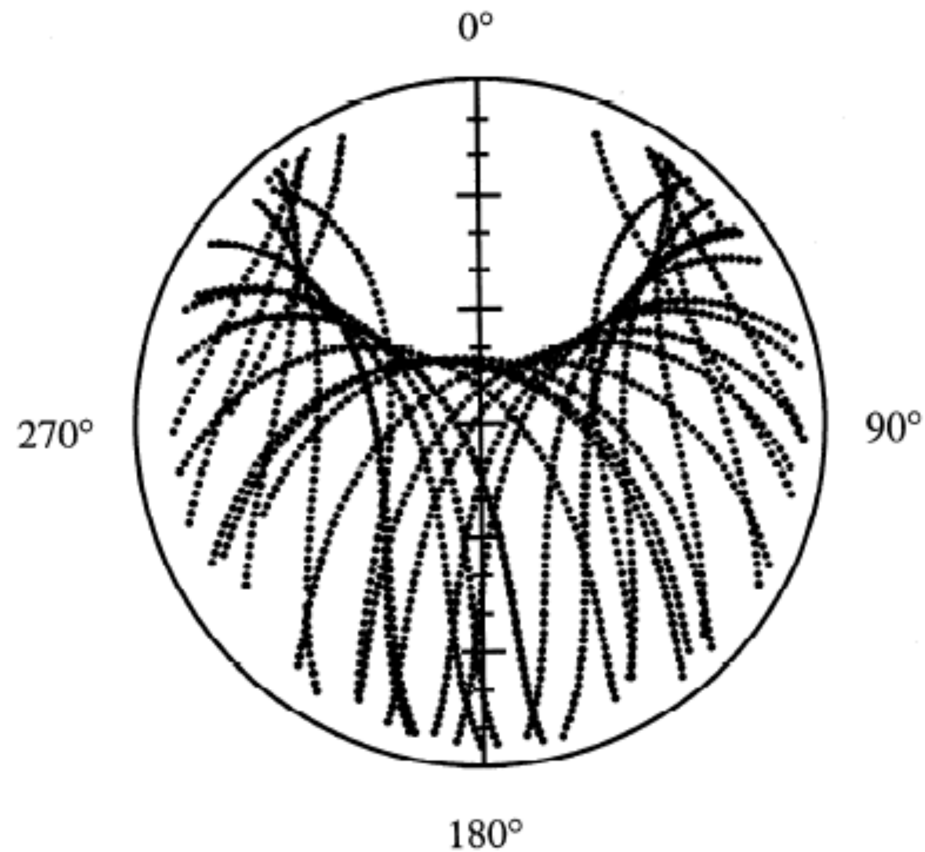
- Six different orbital planes each with five satellites (nominal)
- Within each orbital plane, only reference anomalies differ
- Inclination of all orbits ~55°
- Nearly circular ($e \leq 0.02$)
- Semimajor axis ~26400 km
- Geosynchronous



Wikipedia

GPS Satellite Tracks

- Due to the 55° inclination of all orbits, satellite tracks as seen from ground have "hole"
- Right: Ground track for 24 hours for site with latitude 43° N



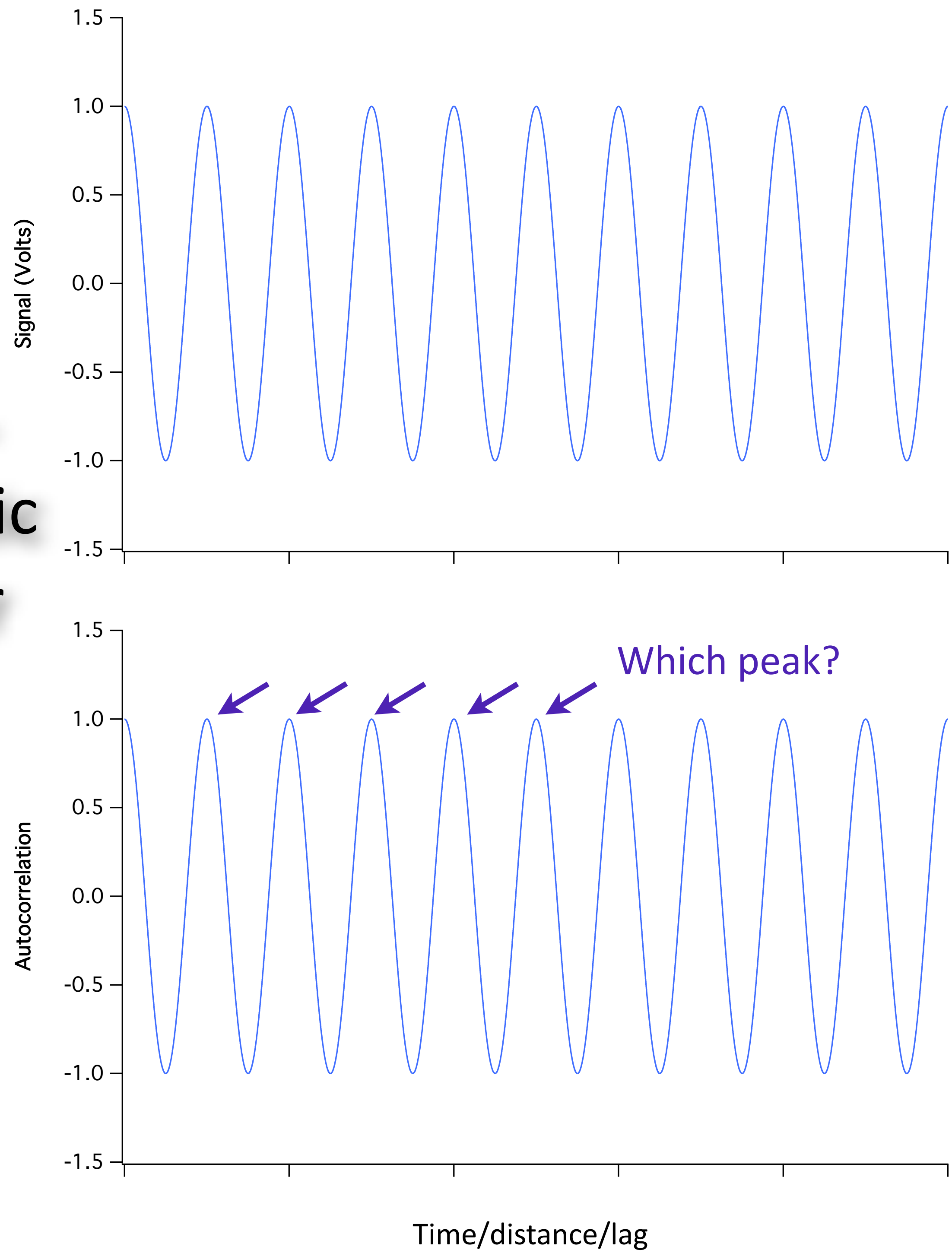
GPS Signals

- GPS satellite transmit signals at two L-band carrier frequencies:
  - L1 f =1575.42 MHz λ = 190 mm
  - L2 f =1227.60 MHz λ = 244 mm
- Both frequencies are integer multiples of GPS fundamental frequency of 10.23 MHz

GPS Signals

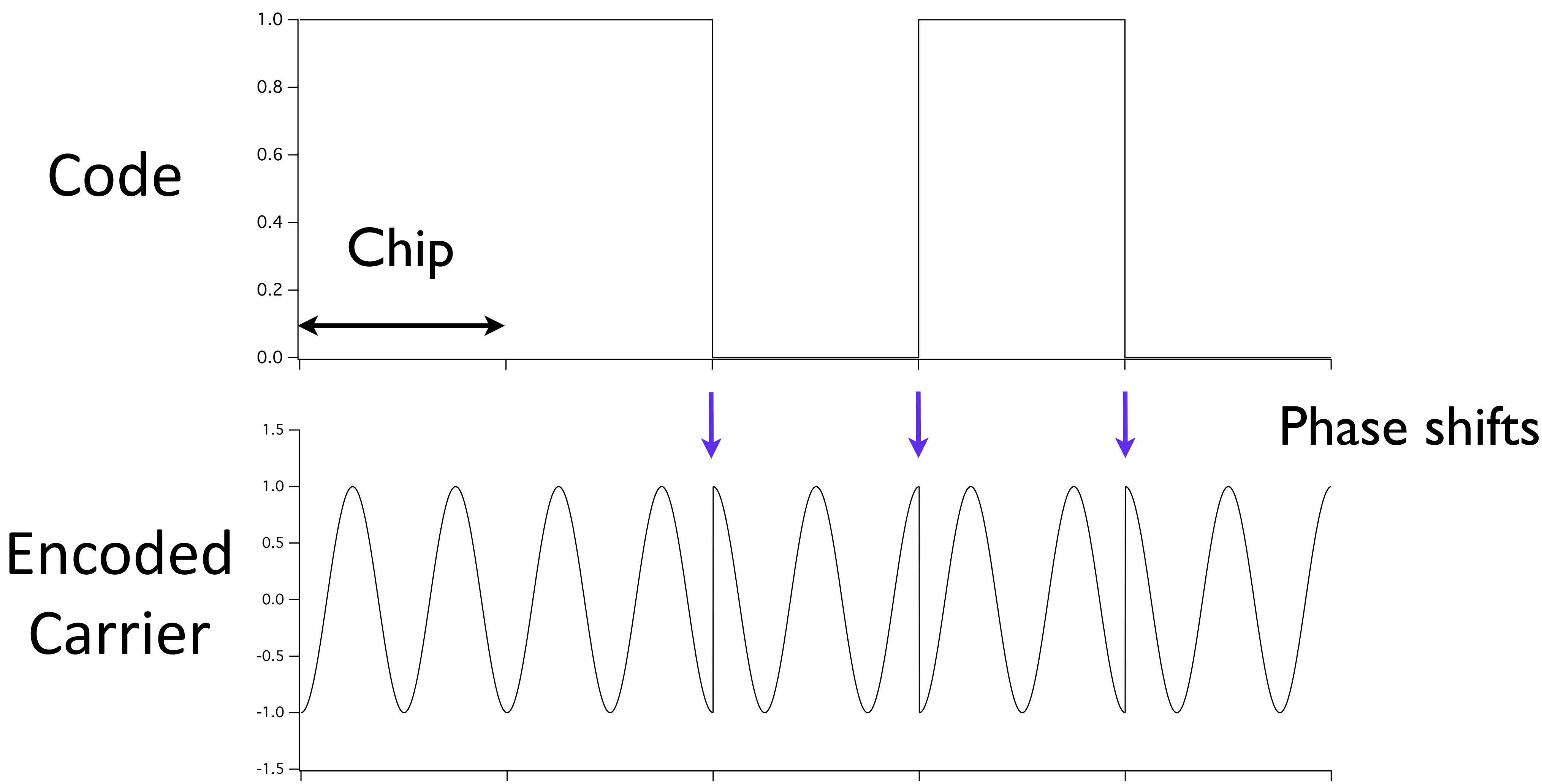
- Both L1 and L2 signals are *encoded*
- The encoding is achieved by shifting the phase of the signal by 180° (binary phase shift keying or bi-phase modulation)
- The code is thus represented as a binary pulse (0 or 1)

Mono-chromatic (carrier only)



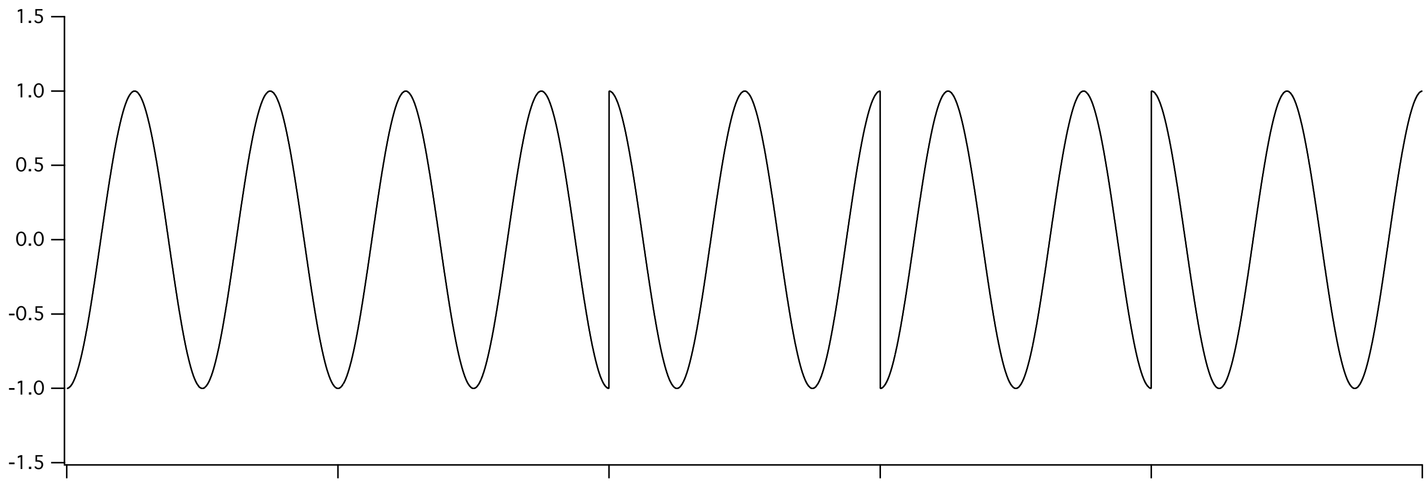
Signal (transmission)

Autocorrelation (detection)

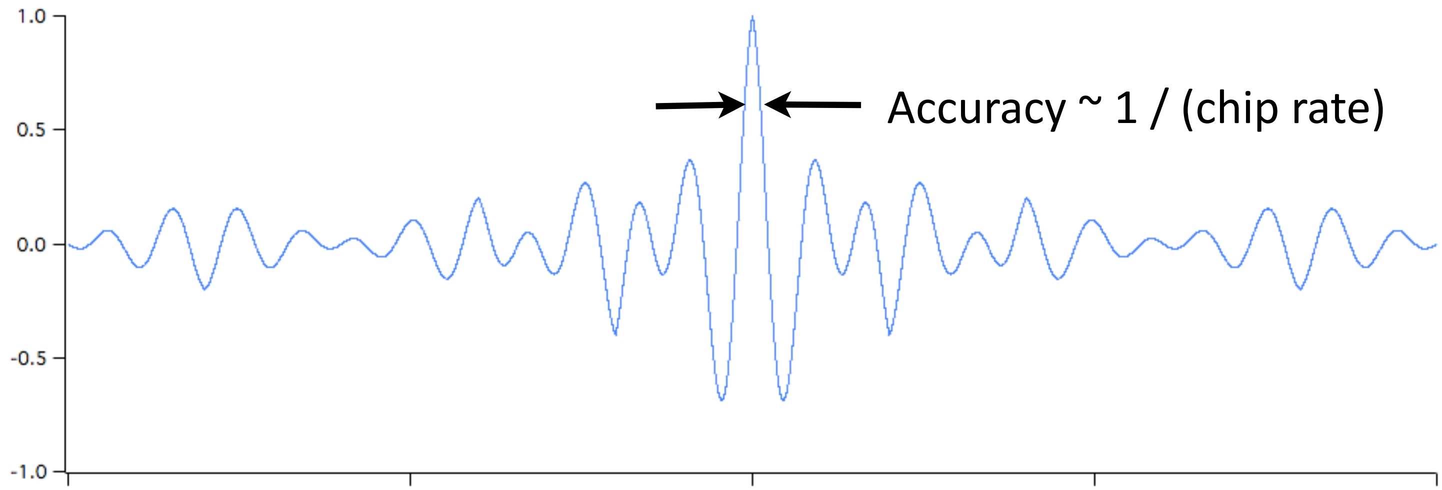


Code
Chip
1.0
0.8
0.6
0.4
0.2
0.0
Phase shifts
Encoded
Carrier
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5

Encoded Carrier



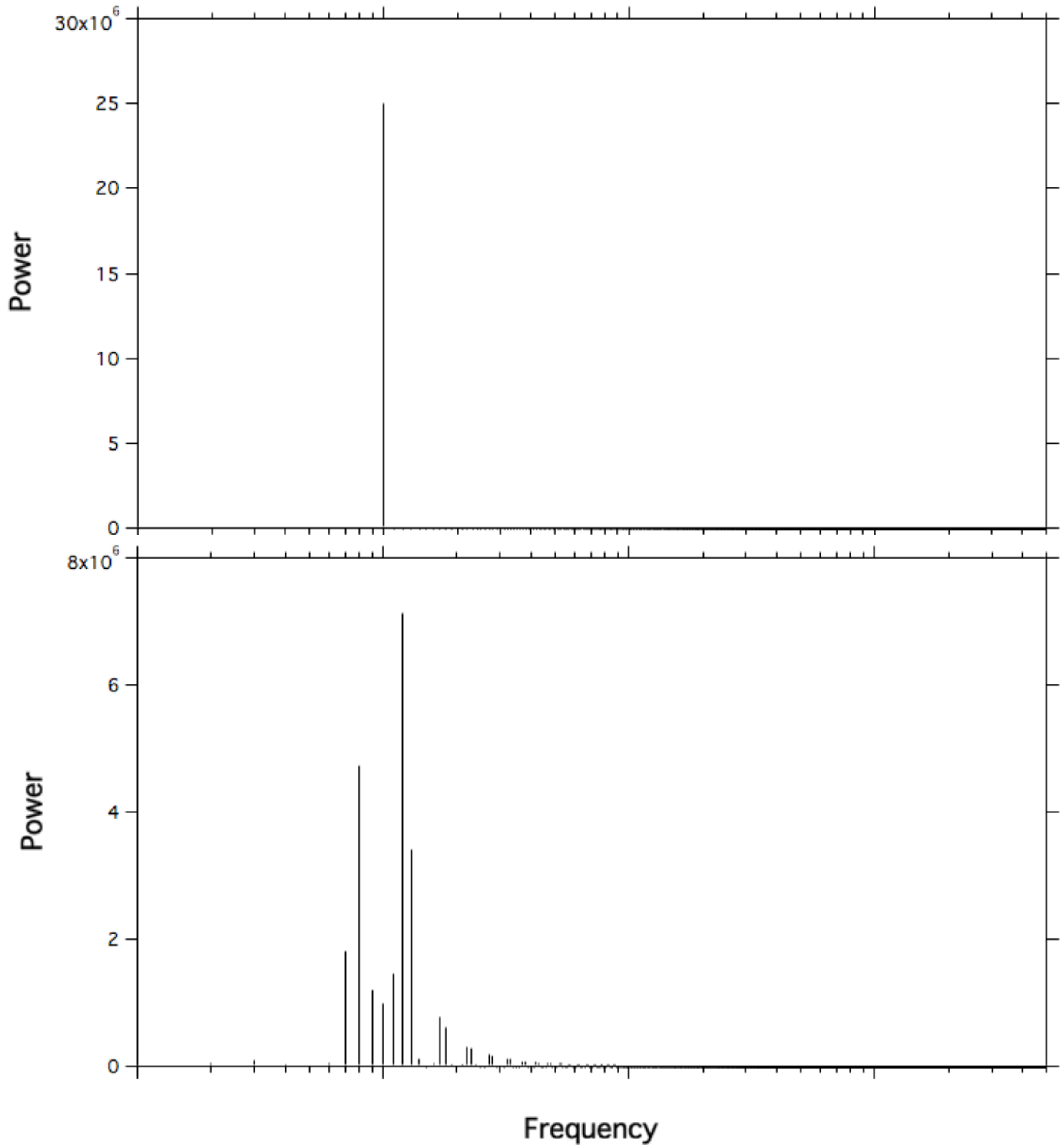
Auto-correlation



Spread-spectrum signal

Carrier

Encoded
Carrier



GPS Signals

- Spread-spectrum encoding for GPS enables a receiver to track multiple satellites simultaneously using the same frequency
- This scheme is also known as Code-Division Multiple Access (CDMA): Multiple signals sharing the same frequency channel with minimum interference between signals
- CDMA underlies mobile phone technology, wherein users share a frequency band but transmit and receive multiple signals

GPS Signal Codes

- Coarse acquisition code (C/A code): Chip rate 1.023 MHz
- Precise positioning code (P code): Chip rate 10.23 MHz
- Y-code (Anti-spoofing, classified): Chip rate 10.23 MHz
- D-code: 50 Hz navigation code

P and C/A Codes

- P-code is 37 weeks long (2.3×10^{14} bits) and then repeats
- Each SV uses the same P-code, shifted by one week
- Pseudorandom, orthogonal
- The SVs are identified by their pseudorandom noise sequence number (PRN)
- C/A code repeats every 1023 bits (1 ms)

Accuracy and chip rate

- D-code: 50 Hz → 5950 km
- C/A code: 1.023 MHz → 293 m
- P-code: 1.023 MHz → 29.3 m

GPS Signals

L_1 $$S_1^p(t) = A_P P^p(t) D^p(t) \cos 2\pi f_1 t + A_C C^p(t) D^p(t) \sin 2\pi f_1 t$$

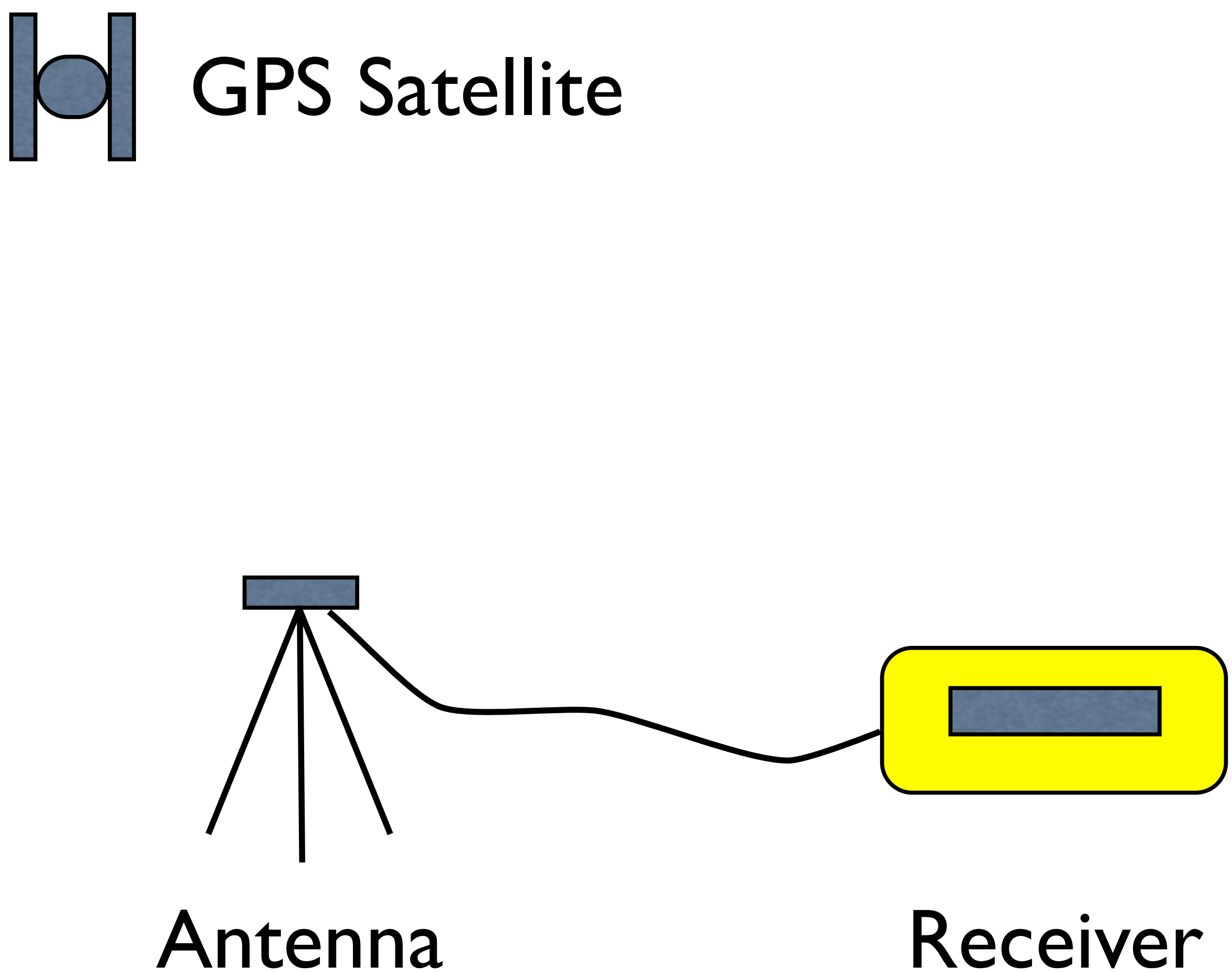
L_2 $$S_2^p(t) = B_P P^p(t) D^p(t) \cos 2\pi f_2 t$$

$S_k^p(t)$	L_k signal for SV p
$A_P,\ B_P\ A_C$	Signal strengths for P, C/A
$D^p(t)$	Navigation data stream
$P^p(t)$	P-code
$C^p(t)$	C/A-code

GPS Modernization

- Civilian (i.e., C/A) codes on L2
- C/A code on third carrier (L5, 1176.45 MHz)
- M-code: Military anti-jamming, autonomous

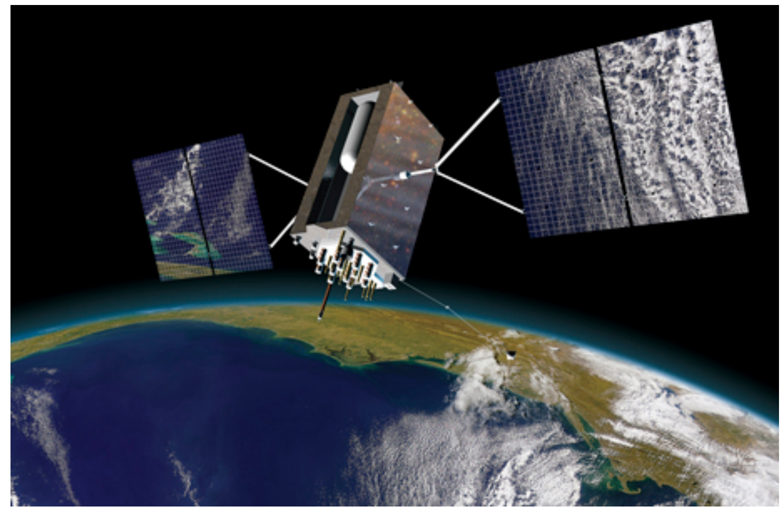
Satellite Acquisition and Tracking



GPS Satellites

Block I
(inactive)

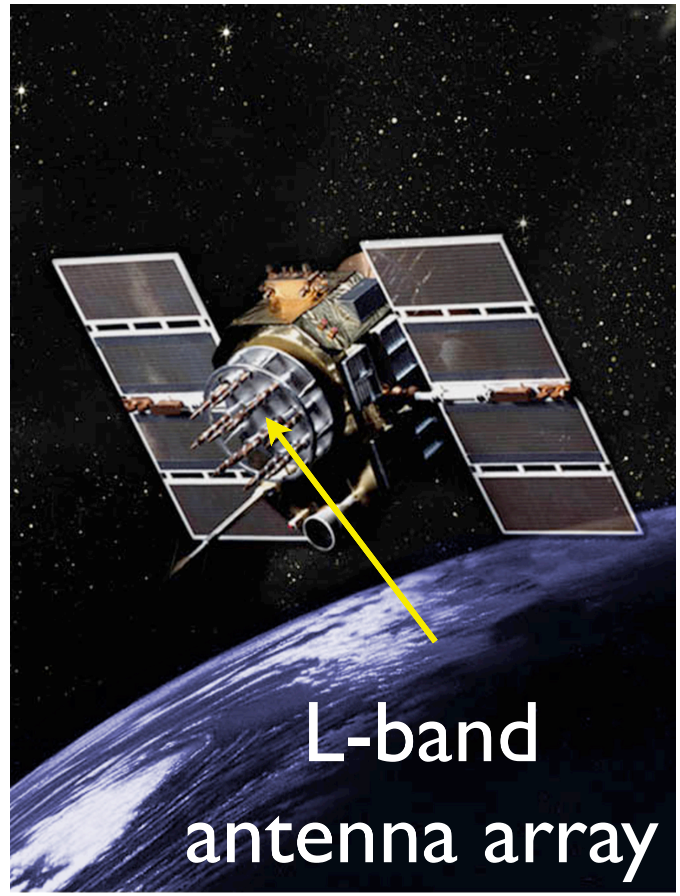
Block II



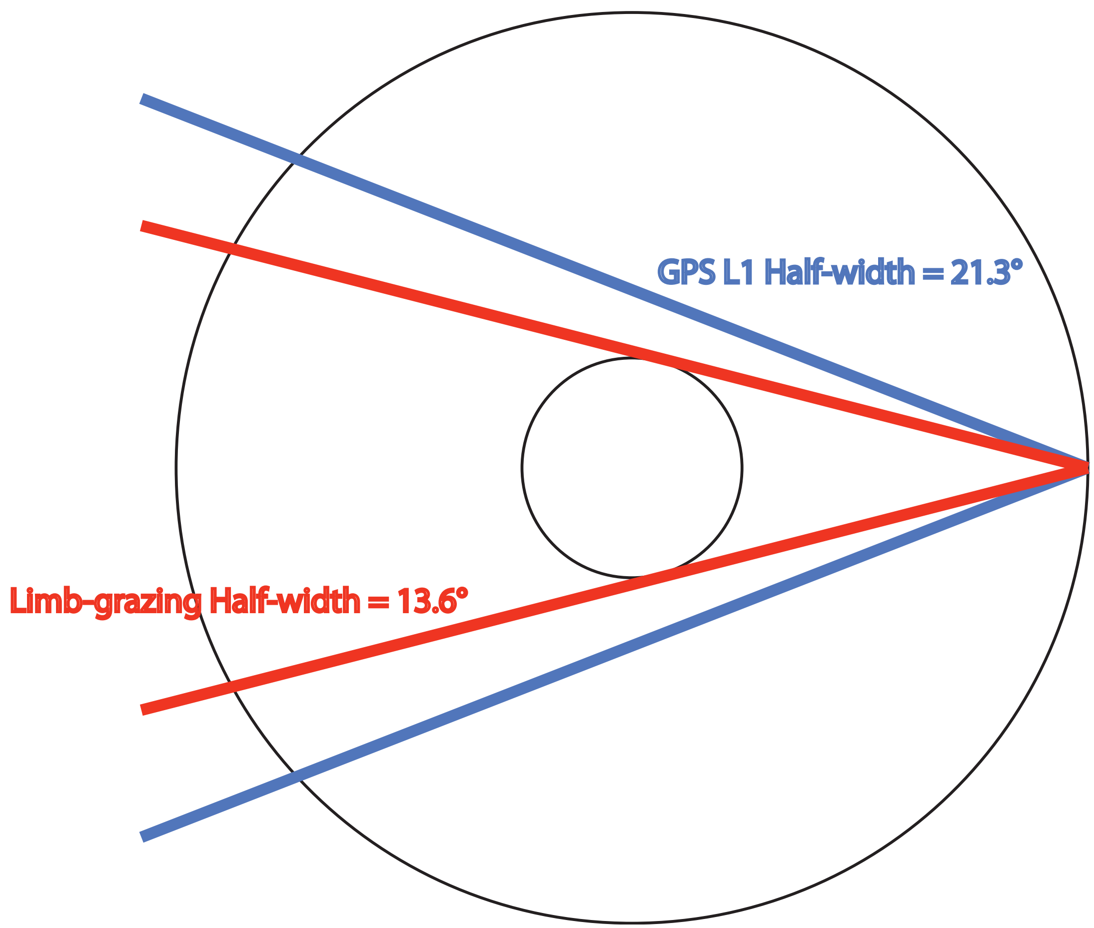
Block III
(future)

GPS Satellite Transmission

- L-band antenna array always points towards center of Earth
- Angular half-width of transmitting beam is 21.3^o at L1, 23.4^o at L2



GPS Satellite Transmission



GPS Satellite Frequency Standards

- As discussed earlier in course, “clocks” are highly accurate frequency standards
- The fundamental GPS frequency is 10.23 MHz
- Clock accuracy (“stability”) is measured as $\frac{\sigma_f}{f_\circ} = \frac{\sigma_t}{T}$
- Typical stability for GPS onboard frequency standards over 24 hours:
    - Rb: 10^{-13} (10 nsec per day)
    - Cs: 10^{-14} (1 nsec per day)
- RINEX broadcast orbit files also provide polynomial corrections to satellite clock

Relativistic clock corrections

- Gravitational redshift:
  - Clocks in different gravitational potentials run at different rates: $\Delta f \simeq \frac{\Delta\Phi}{c^2}$
  - GPS clocks appear to run faster
  - GPS compensates by setting the 10.23 MHz clocks at the factory to 10.229 999 999 543 MHz
- Impact of eccentricity:
  - Clock rate depends on speed in satellite orbit
  - Satellite clock correction $\Delta t_r = -\frac{2\sqrt{GMa}}{c^2} e \sin E = -2\frac{\vec{v}\cdot\vec{r}}{c^2}$
  - This correction can be ~45 nsec

GPS Satellite clock corrections

- The ground segment of the Global Positioning System is used to calculate satellite clock errors
- These errors are modeled as second-order polynomials in time, and uploaded to the GPS satellites
- The GPS satellites broadcast the clock-correction coefficients
- These are the first line of each RINEX data block in the broadcast orbit file

GPS Satellite clock corrections

- The satellite clock correction must include the (eccentricity) relativistic correction also
- The satellite clock correction is therefore

$$\delta^s(t) = a_0 + a_1(t - t_c) + a_2(t - t_c)^2 + \Delta t_r$$

- t_c is the “time of clock” (see RINEX documentation)

Summary: Pseudorange model

- The pseudorange model (Class 1) was

$$\rho(t) = |\vec{x}^s(t-\tau) - \vec{x}_r(t)| + c(\delta_r - \delta^s)$$

- In Class 2, we developed the expression for the satellite position vector
- In this class, we presented the satellite clock correction
- The remaining unknown parameters are:
  - The receiver position vector (3 unknowns: x, y, z)
  - The receiver clock error (1 unknown)

Least-squares overview

- We'll review linear least squares, which we'll use to estimate the unknown parameters
- This is the class of problems in which the model can be written as:

$$y = Ax + \epsilon$$

- Here, y is an $n \times 1$ vector of observations, x is an $m \times 1$ vector of parameters, ϵ is an $n \times 1$ vector of errors, and the $n \times m$ matrix A is the *design matrix* or *partials matrix*

Linearization

- Often our problem will be of the more general and possibly nonlinear form

$$y = f(x) + \epsilon$$

- $f(x)$ is a *vector* of functions
- In this case we linearize around the prior value $x_\circ$
- The design matrix is $A = \left.\frac{\partial f}{\partial x}\right|_{x=x_\circ}$
- And the linearized observation equation

$$\Delta y = A\Delta x + \epsilon$$

- Here $\Delta y = y - f(x_\circ)$ is the vector of *prefit residuals* and the *parameter adjustments* are $\Delta x = x - x_\circ$

Linearization

- Recall is $f(x)$ a vector
- Then really

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

- Then $A = \frac{\partial f}{\partial x}$ is shorthand for $A_{ij} = \frac{\partial f_i}{\partial x_j}$

Least-squares solution

- Given that the errors are unknown, there is no unique solution (value for the parameters) that satifies the observation equation
- Instead, we look for a solution that minimizes the sum of the squared errors, $\epsilon^T \epsilon$
- This solution for the adjustments is

$$\Delta\hat{x} = \left(A^T \Lambda_\epsilon^{-1} A\right)^{-1} A^T \Lambda_\epsilon^{-1} \Delta y$$

- Λ_ϵ is the covariance matrix of the errors (assumed known):

$$\Lambda_\epsilon = \langle \epsilon \epsilon^T \rangle$$

- The least squares estimate of the parameters is $\hat{x} = x_\circ + \Delta\hat{x}$

Data and parameter uncertainties

- The covariance matrix of the errors in the parameter estimates is $\Lambda_x = \left(A^T \Lambda_\epsilon^{-1} A\right)^{-1}$
- The data error covariance matrix Λ_ϵ is usually taken to be diagonal with $[\Lambda_\epsilon]_{ij} = \sigma_i^2 \delta_{ij}$
- In the absence of better information we often take $\Lambda_\epsilon = \sigma^2 I$
- In this case $\Delta\hat{x} = \left(A^T A\right)^{-1} A^T \Delta y$ and $\Lambda_x = \sigma^2 \left(A^T A\right)^{-1}$

Fit Statistics

- Postfit residuals: $\hat{\epsilon} = y - f(\hat{x})$
- Normalized χ^2: $\chi^2 = \left(\frac{1}{n-m}\right)\sum_{i=1}^{n}\frac{\hat{\epsilon}_i^2}{\sigma_i^2}$
- Normalized root-mean-square residual:

$$NRMS = \sqrt{\chi^2}$$

- Weighted root-mean-square residual:

$$WRMS = \sqrt{\left(\frac{n}{n-m}\right)\frac{\sum_{i=1}^{n}\hat{\epsilon}_i^2/\sigma_i^2}{\sum_{i=1}^{n}1/\sigma_i^2}}$$

Fit statistics

- Plot postfit residuals to look for systematic error(s)
- NRMS: Nominal value of 1. Significantly greater than one may indicate systematic error(s) or underestimate of sigmas
- NRMS is often used to scale sigmas
- NRMS significantly less than one may indicate overestimation of sigmas or over-parametrization
- If sigma is unknown, can assume sigma of one and scale all uncertainties by NRMS.